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Solution of the Volterra Integral Form of the Lane-Emden Equation Using the Variational Iteration Method

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ABSTRACT
The Lane-Emden equation plays a pivotal role in modeling astrophysical and fluid dynamic phenomena, such as stellar structures and isothermal gas spheres. Solving this equation poses significant challenges due to its nonlinearity and singularities. While existing literature has addressed these models using the Adomian Decomposition Method, such frameworks require the complex and computationally intensive formulation of Adomian polynomials. This research presents practical ways of applying the Variational Iteration Method as a semi-analytic approach to address these challenges. By obtaining equivalent Volterra integral forms of the Lane-Emden equation and systematically applying Variational Iteration Method, efficient approximate solutions were obtained for five different problems across various parameter cases of shape factor k at $k = 1, 2$ and 3. A favourable series solution was obtained for each of the five problems, and each was independently verified against its closed form exact solution by direct substitution into the given equation and initial conditions. Comparison with these exact solutions showed strong agreement, with absolute errors as small as 10^{-7} to 10^{-8} for the polynomial case, and 10^{-3} to 10^{-5} for the transcendental cases over an identified domain of validity for each problem. These approaches contribute to advancing mathematical modeling in astrophysics and provide a good framework for solving nonlinear differential equations, as shown with the tested problems.

Keywords: Astrophysical Modelling, Lane-Emden Equation, Semi-Analytical Methods, VIM, Volterra Integral Form

INTRODUCTION

Many scientific problems related to physics, astrophysics and stellar structures are modeled to the Lane-Emden equation of the form

$$y''(x) + \frac{k}{x}y' + f(y(x)) = 0, \qquad k > 0 \qquad (1)$$

Subject to the conditions

$$y(0) = \alpha, \quad y'(0) = 0$$

Where $k > 0$ is a constant and it is called shape factor. Equation (1) is the Lane-Emden equation type that was used to model several mathematical physics and astrophysics problem like thermal behaviour of a spherical cloud of gas and theory of stellar structure. Robert Emden studied behaviour of a spherical cloud of gas under classical thermodynamic laws in 1907. Many scientists have discussed various methods of obtaining the solutions of the Lane-Emden equation, including the adomian decomposition method (ADM) (Wazwaz, 2005; Wazwaz et al., 2013), a modified decomposition method (Roul et al., 2019), the local variational iteration method (VIM) (Wang et al., 2019), the Hermite functions collocation method (Parand et al., 2010), the modified Legendre-spectral method (Adibi & Rismani, 2010), and the Gauss-Jacobi collocation method for solving nonlinear Lane-Emden type equations (Bhrawy & Alofi, 2012). This research wants to explore the practical and ways of applying VIM, a semi analytical technique, for solving the Lane-Emden equation. The method is noted for its simplicity and efficiency in handling nonlinear problems.

Basic Mathematical Definitions

Integral Equation

Integral equation is an equation that the function $y(x)$ we want to obtain its solution is also appears inside the integral sign as shown in the equation (2) below

$$y(x) = f(x) + \int_{\alpha(x)}^{\beta(x)} K(x,t)y(t)\,dt \qquad (2)$$

Where $K(x,t)$ is known as the kernel of the equation, its play a central role that specifies the relationship between x and t which is sometimes symmetric or non-symmetric and $\alpha(x)$and$\beta(x)$are lower and upper limits of the integration (Wazwaz, 2015).

Fredholm Integral Equations

The Fredholm integral equations, is one of integral equation where $\alpha(x) = a\beta(x) = b$ are constants, consider equation (3) below:

$$y(x) = f(x) + \lambda \int_b^a K(x,t)y(t)\,dt,\ a \le x,\ t \le b \qquad (3)$$

Where the kernel $K(x,t)$ and the function $f(x)$, is defined and λ is a parameter (Wazwaz, 2015).

Volterra Linear Integral Equations

A Volterra integral equation is a type of integral equation where the upper limit of integration is a variable, often denoted by x. The general form of a Volterra linear integral equation is:

$$y(x) = f(x) + \lambda \int_a^x K(x,t)y(t)\,dt, \qquad (4)$$

Where the unknown function $y(x)$ under the integral sign occurs linearly. Note that (4) is another special way of written Fredholm integral equation when the kernel $K(x, t)$ vanishes at $t > x$, $x \in [a, b]$.

Correctional Functional

The correctional functional is an iterative construct used in the VIM to iteratively correct an initial approximation of the solution to a given problem. It is expressed in the form:

$$u_{n+1}(x) = u_n(x) + \int_0^x \lambda(t)\,R_n(t)dt \qquad (5)$$

Where $u_{n+1}(x)$ is the updated approximation, $\lambda(t)$ is the Lagrange multiplier determined through variational principle, and $R_n(t)$ is the residual error of the nth approximation. The $R_n(t)$ represents the error or deviation when the current approximation $u_n(x)$is substituted into the given equation. It also measures how far $u_n(x)$ is from being the solution of the given problem. Correctional functional acts as a correction term, modifying the initial condition in a way that systematically reduces the residual error iteratively between the approximate solution and ensuring convergence to the exact solution (Abdulla-Al-Mamun, 2019).

Literature Review

Research on the VIM has progressed from its original theoretical foundations toward an increasingly wide range of applications, including systems of nonlinear differential equations, fractional and integro-differential problems, and several forms of the Lane-Emden equation. The studies most directly relevant to the present work are reviewed below. Tripathi (2012) highlights the effectiveness of the method for nonlinear differential equations of fractional order. This article emphasizes the use of Hermits functions collocation method to derive nonperturbative approximate solutions for Lane-Emden type equations, which are significant in astrophysical contexts. The study also introduces a modified decomposition method that enhances the analytic treatment of differential equations, setting a foundational understanding of how VIM can be adapted for complex nonlinear scenarios.

Following this, Rangkuti and Noorani (2018) expand on the versatility of VIM by coupling it with Taylor series and small term methodologies to tackle delay differential equations (DDEs). Their investigation illustrates the adaptability of VIM beyond traditional applications, showcasing its utility in various fields such as biochemistry and control systems. The authors provide analytical solutions through a systematic approach using general Lagrange multipliers, thus reinforcing the potential of VIM in achieving exact solutions even in the presence of delays an aspect that may have implications under certain conditions for solving the Lane-Emden equation.

Wang et al., (2019) further contribute to the discourse by proposing a local VIM that simplifies the computational process for nonlinear science problems. Their derivation of an analytical iteration formula from first-order nonlinear differential equations, coupled with the use of Chebyshev polynomials and collocation methods, illustrates a significant advancement in numerical algorithms. The authors emphasize the efficiency of their approach, particularly in leveraging GPU acceleration for complex nonlinear problems, which includes applications to equations like the Chandrasekhar equation. This work not only demonstrates the practical applicability of VIM in solving nonlinear systems but also underscores the importance of numerical methods in advancing our understanding of nonlinear phenomena.

Wazwaz (2005) utilized the ADM to derive analytical solutions for time dependent Emden Fowler type equations and wave type equations exhibiting singular behaviour. This singular global method offers a robust framework for effectively addressing coordinate singularities that arise in differential equations defined over semi-infinite domains, facilitating more reliable and computational series solutions. By defining a modified differential operator that incorporates the singular terms directly, the approach successfully bypasses the need for complex coordinate transformations. This operator design enables an accurate approximation of the exact solutions while preserving the geometric shape factor coefficients inherent to the physical model.

Mendelzweig and Tabakin (2001) employed a quasi-linearization technique to address the Lane-Emden equations. This approach approximates the solution of the nonlinear differential equation by treating the nonlinear terms as perturbations of linear terms, thereby providing a different perspective compared to traditional methods. Notably, this technique does not rely on the presence of a small parameter, which distinguishes it from other theoretical frameworks. Aslanov (2009) investigated individual initial value problems and expanded a decomposition method to accommodate a variety of Emden-Fowler like equations. Their work demonstrated the reliability of this system while also addressing specific cases of the equation as illustrative examples. Remarkably, the solutions derived from this method take the form of a convergent series solution, which allow for the possibility of obtaining solutions to Emden Fowler type equations even in scenarios where the function (y) is not analytic at the origin. This overcomes the challenges posed by singular points at $x = 0$ and discontinuities, thereby enhancing the versatility of the approach.

Mechanika, (2016) demonstrate variational iteration method for tackling linear and nonlinear two. Abbasbandy and Ghehsareh (2010) demonstrate the effectiveness of variational iteration method in solving the integro differential parabolic problem with integral conditions, its convergence under certain conditions and illustrate the efficiency of their method with numerical examples.

Abdulla –Al –Mamun, (2019) used VIM to solve Volterra's integro-differential equations. The VIM is an effective and accurate tool for addressing a variety of non-linear problems with fast converging approximations. For linear problems, the Lagrange multiplier can be pinpointed exactly so that a single step of iteration suffices to obtain an exact solution. It is obvious that the Lagrange multiplier helps reduce iteration on the integral operator, thereby reducing computational time. Karimi et al, (2020) introduced an iterative spectral method designed to approximate the solutions of Lane-Emden equations which involves extending the Legendre wavelet, which is suitable for any time interval, as an initial step. Subsequently, obtain the Gauss-Legendre collection points of the extended Legendre wavelet. With this novel methodology, the iterative spectral technique transforms the differential equation into a system of algebraic equations, effectively reducing computational costs. By solving these derived algebraic equations and attain a precise and reliable approximate solution for the given Lane-Emden equation.

A closely related study by Wazwaz et al. (2013) also addressed the Volterra integral form of the Lane-Emden equation, obtaining series solutions with the Adomian decomposition method for problems with prescribed initial and boundary values. The present work addresses the same underlying integral reformulation but applies the VIM instead, which constructs successive approximations through a Lagrange-multiplier correction functional rather than through decomposition of the nonlinearity into Adomian polynomials. This distinction is the primary point of departure, and situates the present contribution as an alternative solution technique for the same class of Volterra-reformulated Lane-Emden problems rather than as an extension of the decomposition-based approach.

Variational Iteration Method (VIM)

To implement the method of VIM we need to write down a correction functional of the solution, incorporating a Lagrange multiplier that is identified through Variational theory.

Let consider the differential equation of the form:

$Lu + Nu = g(t)$(6)

Where *L* and *N*are linear and nonlinear operators respectively (He, 1999), the correction functional for equation (6) is in the form:

$u_{n+1}(x) = u_n(x) + \int_0^x \lambda(x,t)\left(Lu_n(t) + N\tilde{u}_n(t) - g(t)\right)dt$ (7)

$\lambda(x,t)$: Lagrange multiplier

u_n: nth approximation

By identifying the Lagrange multipliers and applying them iteratively, we generate successive approximations converging to the actual solution.

Determination of Lagrange Multiplier

Consider the differential equation and correction functional above of the form:

$Lu + Nu = g(t)$

Then, correction functional is

$u_{n+1}(x) = u_n(x) + \int_0^x \lambda(x,t)\left(Lu_n(t) + N\tilde{u}_n(t) - g(t)\right)dt$

Where $\lambda(x,t)$ is general Lagrange's multiplier, it may be constant or function, and $\tilde{u}_n(t)$ behaves as a constant, therefore $\delta\tilde{u}_n = 0$, where δ is known as variational derivative.

Thorough variational theory we can be able to identified the Lagrange multiplier $\lambda(x,t)$ optimally. Variational theory, the extremum of a function is obtained by setting the first variation of the function to zero. i.e $\delta u_{n+1} = 0$. This leads to necessary conditions often involving boundary terms and integral terms. Since the variation $\delta u_n(t)$ is arbitrary then each term must be independently vanished to ensure the stationarity of the functional. The process reflects the variational derivative approach, where terms from both boundary and integrals are evaluated to derive necessary conditions for optimality. (Santambrogio, 2023). Taking the variation derivative of (7) with respect to the independent variable $u_n(t)$ yields:

$\delta u_{n+1} = \delta u_n + \delta\left(\int_0^x \lambda(t)\left(Lu_n(t) - g(t)\right)dt\right)$ (18)

If non-homogeneous term is equal to zero, then equation (8) reduces to;

$\delta u_{n+1} = \delta u_n + \delta\left(\int_0^x \lambda(t)\left(Lu_n(t)\right)dt\right)$

Note that $Lu_n(t)$ can be equal to ${u'}_n(t)$ or ${u''}_n(t)$ …….${u^n}_n(t)$

To determine the Lagrange multiplier $\lambda(t)$ let consider the term

When $Lu_n(t) = {u'}_n(t)$

Apply Integration by parts,

$\int_0^x \lambda(t)\,{u'}_n(t)dt = \lambda(t)u_n(t) - \int_0^x \lambda'(t)\,u_n(t)dt$ (10)

Similarly, if $Lu_n(t) = {u''}_n(t)$ and $Lu_n(t) = {u'''}_n(t)$ we have;

$\int_0^x \lambda(t)\,{u''}_n(t)dt = \lambda(t){u'}_n(t) - \lambda'(t)u_n(t) + \int_0^x \lambda''(t)\,u_n(t)dt$ (11)

$\int_0^x \lambda(t)\,{u'''}_n(t)dt = \lambda(t){u''}_n(t) - \lambda'(t){u'}_n(t) + \lambda''(t)u_n(t) - \int_0^x \lambda'''(t)\,u_n(t)d$ (12)

And so on.

Substitute (10) into (9) when $\lambda(t)Lu_n(t) = \lambda(t){u'}_n(t)$ we have;

$\delta u_{n+1} = \delta u_n + \delta\left(\lambda(t)u_n(t) - \int_0^x \lambda'(t)\,u_n(t)dt\right)$

$\delta u_{n+1} = \delta u_n + \lambda(t)\delta u_n(t) - \int_0^x \lambda'(t)\,\delta u_n(t)dt$

Which can be simplify as

$\delta u_{n+1} = \delta u_n(1 + \lambda|_{t=x}) - \int_0^x \lambda'(t)\,\delta u_n dt \;\; \delta u_n > 0$ (13)

The extremum condition of u_{n+1} requires that $\delta u_{n+1} = 0$ this means that the LHS is zero and the result at RHS should be zero as well. This yields the stationary conditions:

$1 + \lambda|_{t=x} = 0, \quad \lambda'|_{t=x} = 0$

Thus, $\lambda(x,t) = -1$

And so on, generally for the $nth\ order$ ODE the Lagrange multiplier λ can be concluded to have the formula $\lambda(x,t) = (-1)^n \frac{(t-x)^{n-1}}{(n-1)!}$

MATERIALS AND METHODS

Methodology

Transformation into Volterra Integral Form when $k > 1$

We are transforming Lane-Emden equation of the form (1) into Volterra integral form of

$y(x) = \alpha - \frac{1}{k-1}\int_0^x t\left(1 - \frac{t^{k-1}}{x^{k-1}}\right)f(y(t)\,dt$ (14)

Let consider the Lane-Emden Equation

$y'' + \frac{k}{x}y' = -f(y)$

Using the method of integrating factor, I.F

Let's $p(x) = \frac{k}{x}$

$I.F = e^{\int p(x)dx} = e^{\int \frac{k}{x}dx} = e^{klnx} = x^k$

The general solution for the method of integrating factor is given as

$y'.(I.F) = \int -f\big(y(t)\big)(I.F)dt$

$y'' + \frac{k}{x}y' = -f(y)$

$x^k y' = \int_0^x -f\big(y(t)\big)t^k dt$

$y' = -\frac{1}{x^k}\int_0^x t^k f\big(y(t)\big)dt$ (15)

Integrating (15) with respect to x then, we have

$y|_0^x = -\int_0^x \left[\frac{1}{x^k}\int_0^x t^k f\big(y(t)\big)dt\right]dx$

$[y]_0^x = -\int_0^x \int_0^x t^k f\big(y(t)\big)dt\,.\frac{1}{x^k}dx$

Replace $\left(\frac{1}{x^k}\right)$ with $\frac{1}{k-1}d\left(\frac{1}{x^{k-1}}\right)$

$y(x) - y(0) = \int_0^x \left[\int_0^x t^k f\big(y(t)\big)dt\,.\frac{1}{k-1}d\left(\frac{1}{x^{k-1}}\right)\right]dx$

$y(x) - \alpha = \frac{1}{k-1}\int_0^x \left[\int_0^x t^k f\big(y(t)\big)dt\,.d\left(\frac{1}{x^{k-1}}\right)\right]dx$

Using integration by part; by letting $u = \int_0^x t^k f\big(y(t)\big)dt\,, dv = d\left(\frac{1}{x^{k-1}}\right)$ we have;

$y(x) - \alpha = \frac{1}{k-1}\left[\left[\int_0^x t^k f\big(y(t)\big)dt\,.\frac{1}{x^{k-1}}\right]_0^x - \int_0^x \frac{1}{x^{k-1}}.d\left(\int_0^x t^k f(y)dt\right)\right]$

$y(x) - \alpha = \frac{1}{k-1}\left[\left[\int_0^x t^k f\big(y(t)\big)dt\,.\frac{1}{x^{k-1}}\right]_0^x - \int_0^x \frac{1}{x^{k-1}}.x^k f\big(y(x)\big)dx\right]$

$y(x) - \alpha = \frac{1}{k-1}\left[\int_0^x t^k f\big(y(t)\big)dt.\frac{1}{x^{k-1}} - \int_0^x tf\big(y(t)\big)dt\right]$

$y(x) - \alpha = \frac{1}{k-1}\int_0^x \left(\frac{t^k}{x^{k-1}} - t\right)f(y(t)\,dt$

Further simplification we have;

$y(x) = \alpha - \frac{1}{k-1}\int_0^x t\left(1 - \frac{t^{k-1}}{x^{k-1}}\right)f(y(t)\,dt, k > 1$ (16)

Equation (16) is now the Volterra Integral form of the Lane-Emden equation when *k>1*. To satisfy that equation (16) is the solution of equation (14), let obtain the first and second derivative of equation (16) and substitute then back into (14).

$y(x) = \alpha - \frac{1}{k-1}\int_0^x t\left(1 - \frac{t^{k-1}}{x^{k-1}}\right)f\big(y(t)\big)dt$

By Leibniz rule to differentiate integral equation (16) of the form

$\frac{d}{dx}\int_{\gamma(x)}^{\beta(x)} G(x,t)dt = G\big(x,\beta(x)\big)\frac{d\beta(x)}{dx} - G\big(x,\gamma(x)\big)\frac{d\gamma(x)}{dx} + \int_{\gamma(x)}^{\beta(x)} \frac{\partial}{\partial x}G(x,t)dt$ (17)

(Talvila, 2001)

$\frac{d}{dx}y(x) = \frac{d}{dx}\left(\alpha - \frac{1}{k-1}\int_0^x t\left(1 - \frac{t^{k-1}}{x^{k-1}}\right)f\big(y(t)\big)dt\right)$ (18)

By applying (17) where $\beta(x) = x,\ \gamma(x) = 0,\ G(x,t) = t\left(1 - \frac{t^{k-1}}{x^{k-1}}\right) f\big(y(t)\big)$, and $\frac{d\beta(x)}{dx} = 1, \frac{d\gamma(x)}{dx} = 0, \frac{\partial}{\partial x} G(x,t) = t\left(\frac{t^{k-1}}{x^k}\right) f\big(y(t)\big)$ we have;

$$y'(x) = 0 - \frac{1}{k-1}\Big[1(t(1-1) - 0 + (k-1)\int_0^x t\left(\frac{t^{k-1}}{x^k}\right) f\big(y(t)\big)dt\Big]$$

$$y'(x) = -\frac{1}{k-1}\Big[(k-1)\int_0^x \left(\frac{t^k}{x^k}\right) f\big(y(t)\big)dt\Big]$$

$$y'(x) = -\int_0^x \left(\frac{t^k}{x^k}\right) f\big(y(t)\big)dt \qquad (19)$$

Differentiate equation (3.6) we have;

$$\frac{d}{dx}y'(x) = \frac{d}{dx}\Big[-\int_0^x \left(\frac{t^k}{x^k}\right) f\big(y(t)\big)dt\Big]$$

Apply Leibniz again, we have;

$$y''(x) = -\Big[\left(\frac{x^k}{x^k}\right) f\big(y(x)\big) - 0 + (-k)\int_0^x t^k\, x^{-k-1} f\big(y(t)\big)dt\Big]$$

$$y''(x) = -\Big[(1) f\big(y(x)\big) + (-k)\int_0^x \left(\frac{t^k}{x^{k+1}}\right) f\big(y(t)\big)dt\Big]$$

$$y''(x) = -f\big(y(x)\big) + k\int_0^x \left(\frac{t^k}{x^{k+1}}\right) f\big(y(t)\big)dt \qquad (20)$$

Substitute (19) and (20) into generalized lane-Emden equation and the equation $y'' + \frac{k}{x}y' + f(y) = 0$ is satisfied. This shows that the Volterra Integral form (16) is the general solution form of Lane-Emden equation.

Volterra Integral Form of the Lane-Emden Equation for $k = 1$

To obtain its integral form at $k = 1$, we let (16) take the limit of $k \to 1$.

Equation (16) is the Volterra Integral form of Lane-Emden Equation for $k > 1$ then at $k = 1$ is a special case of integral form.

$$y(x) = \alpha - \frac{1}{k-1}\int_0^x t\left(1 - \frac{t^{k-1}}{x^{k-1}}\right) f\big(y(t)\big)\, dt$$

$$\lim_{k\to 1}\left\{y(x) = \alpha + \frac{1}{k-1}\int_0^x t\left(\frac{t^{k-1}}{x^{k-1}} - 1\right) f\big(y(t)\big)\, dt\right\}$$

$$\lim_{k\to 1}\{y(x;k)\} = \alpha + \int_0^x t\left\{\lim_{k\to 1}\frac{1}{k-1}\left(\left(\frac{t}{x}\right)^{k-1} - 1\right)\right\} f\big(y(t)\big)\, dt$$

Taking this term from the integral and obtain the limit at $k \to 1$

$$\lim_{k\to 1}\frac{1}{k-1}\left(\left(\frac{t}{x}\right)^{k-1} - 1\right) = \lim_{k\to 1}\frac{\left(\frac{t}{x}\right)^{k-1} - 1}{k-1} = \frac{0}{0}$$

Then we apply L Hospital's rule of calculus i.e the derivative of quotient independently

$$\lim_{k\to 1}\frac{\left(\frac{t}{x}\right)^{k-1}\ln\frac{t}{x}}{1} = \left(\frac{t}{x}\right)^{1-1}\ln\frac{t}{x} = \left(\frac{t}{x}\right)^{0}\ln\frac{t}{x} = \ln\left(\frac{t}{x}\right)$$

Substitute the obtained result $\ln\left(\frac{t}{x}\right)$ to the integral, we have;

$$y(x) = \alpha + \int_0^x t\ln\left(\frac{t}{x}\right) f(y(t))dt \qquad (21)$$

Equation (21) is the volterra integral of lane Emden equation at $k = 1$

Therefore, (16) and (21) form the volterra integral forms of the Lane-Emden equations. These volterra equations eliminate the singularity at the origin, allowing smooth application of the VIM.

RESULTS AND DISCUSSION

Numerical Experiments

Introduction

We present the numerical practical application of VIM combined with Volterra Integral equivalent to solve different cases of the Lane-Emden equation. The VIM, based on constructing correction functionals using variational theory, has been applied to five different forms of the Lane-Emden equation, each representing a distinct physical interpretation.

Example1

Consider the Lane-Emden type equation with $k = 1$

$$y'' + \frac{1}{x}y' - 4e^{1-2y} = 0\ ,\quad y(0) = 1,\ y'(0) = 0$$

Solution: Let consider the first order Volterra Integro-differential form (19) derived as

$$y'(x) = -\int_0^x \left(\frac{t^k}{x^k}\right) f\big(y(t)\big)dt$$

Put $k = 1$ and $f\big(y(t)\big) = -4e^{1-2y(t)}$ in the above equation, we have

$$y' = 4\int_0^x \frac{t}{x} e^{1-2y(t)}dt$$

By VIM, the correction functional set by equation (7) for the given problem is of the form;

where $\lambda(t) = -1$

$$y_{n+1}(x) = y_n(x) - \int_0^x \left(y_n{}' - 4\int_0^x \frac{t}{x} e^{1-2y_n(t)}dt\right)dx \quad n \geq 0$$

for $n = 0$ and

$$y_1(x) = y_0(x) - \int_0^x \left(y_0{}' - 4\int_0^x \frac{t}{x} e^{1-2y_0(t)}dt\right)dx$$

From the given initial conditions

$$y(0) = 1, \quad y'(0) = 0$$

$$y_1(x) = 1 - \int_0^x \left(0 - 4\int_0^x \frac{t}{x} e^{1-2(1)}dt\right)dx$$

Integrate the above and follow the same procedure for other iterations, the following series approximate solution was obtained.

$$y_1(x) = 1 + \frac{x^2}{e}$$

$$y_2(x) = 1 + \frac{x^2}{e} - \frac{1}{2e^2}x^4 + \frac{2}{9e^3}x^6 - \frac{x^8}{12e^4} + \cdots$$

$$y_3(x) = 1 + x^2e^{-1} - \frac{x^4e^{-2}}{2} + \frac{x^6e^{-3}}{3} - \frac{17x^8e^{-4}}{72} + \cdots$$

Using the initial conditions, the above approximations were obtained for $y(x)$. Since the noise term vanishes in the limit, the exact solution is given as $y(x) = ln(x^2 + e)$, which was verified in this study by direct substitution into the governing equation and initial conditions.

Table 1 compares this exact solution with the VIM approximation $y_n(x), n = 3$, obtained above, evaluated at selected points of x, alongside the resulting absolute error.

Table 1: Comparison of the Exact Solution and the VIM Approximate Solution for Example 1

x	Exact solution	VIM $y_n(x), n = 3$	|Error|
0.1	1.0036720	1.0036720	2.41e-12
0.2	1.0146080	1.0146080	5.15e-10
0.3	1.0325728	1.0325729	8.95e-09
0.4	1.0571935	1.0571936	3.20e-08
0.5	1.0879833	1.0879830	2.29e-07
0.6	1.1243716	1.1243685	3.07e-06
0.7	1.1657355	1.1657171	1.84e-05
0.8	1.2114295	1.2113511	7.84e-05

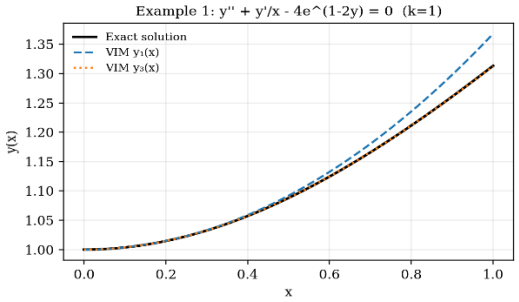

Figure 1: Exact Solution Versus Successive VIM Iterates for Example 1

As shown in Table 1 and Figure 1, the VIM approximation is accurate to within 10^{-4} for x up to about 0.8.

Example 2

Considered the Lane-Emden type with $k = 1$

$y'' + \frac{1}{x}y' + 4(y^2 - 2y^3) = 0\,, \quad y(0) = 1,\ y'(0) = 0$

Solution: Let consider the first order Volterra Integro-differential form (19) derived as

$y'(x) = -\int_0^x \left(\frac{t^k}{x^k}\right) f(y(t))dt$

Put $k = 1$ and $f(y(t)) = 4(y^2 - 2y^3)$ in the above equation, we have

$y' = -4\int_0^x \frac{t}{x}(y^2 - 2y^3)dt$

The correction functional is set by

$y_{n+1}(x) = y_n(x) - \int_0^x \left(y_n{}' + 4\int_0^x \frac{t}{x}(y_n{}^2 - 2y_n{}^3)dt\right)dx \qquad n \geq 0$

for $n = 0$

$y_1(x) = y_0(x) - \int_0^x \left(y_0{}' + 4\int_0^x \frac{t}{x}(y_0{}^2 - 2y_0{}^3)dt\right)dx$

Integrate the above and follow the same procedure for other iterations, the following series approximate solution was obtained.

$y_1(x) = 1 + x^2$

$y_2(x) = 1 + x^2 + x^4 + \frac{5x^6}{9} + \frac{x^8}{8}$

$y_3(x) = 1 + x^2 + x^4 + x^6 + \frac{2x^8}{9} + \frac{307x^{10}}{450} + \frac{581x^{12}}{1296} + \cdots$

$y_4(x) = 1 + x^2 + x^4 + x^6 + x^8 + \frac{221x^{10}}{225} + \frac{1891x^{12}}{2025} + \cdots$

Using the initial conditions, the above approximations were obtained for $y(x)$. Since the noise term vanishes in the limit, the exact solution is given as $y(x) = \frac{1}{1-x^2}$, which was verified in this study by direct substitution into the governing equation and initial conditions.

Table 2 compares this exact solution with the VIM approximation $y_n(x), n = 4$, obtained above, evaluated at selected points of x, alongside the resulting absolute error.

Table 2: Comparison of the Exact Solution and the VIM Approximate Solution for Example 2

x	Exact solution	VIM $y_n(x), n = 4$	|Error|
0.1	1.0101010	1.0101010	1.85e-12
0.2	1.0416667	1.0416667	2.26e-09
0.3	1.0989011	1.0989009	1.93e-07
0.4	1.1904762	1.1904700	6.17e-06
0.5	1.3333333	1.3332184	1.15e-04
0.6	1.5625000	1.5610240	1.48e-03
0.7	1.9607843	1.9460677	1.47e-02
0.8	2.7777778	2.6491536	1.29e-01

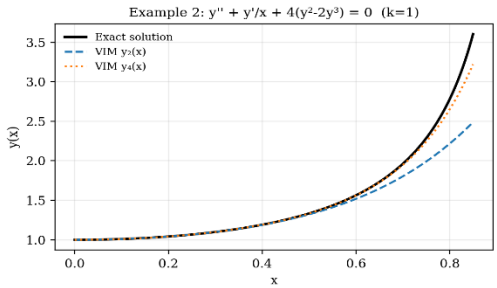

Figure 2: Exact Solution Versus Successive VIM Iterates for Example 2

As shown in Table 2 and Figure 2, the VIM approximation is accurate to within 10^{-4} for x up to about 0.5; beyond this the truncated series departs increasingly from the exact solution as x approaches the pole at x = 1.

Example 3

Considered the Lane-Emden type with $k = 2$

$y'' + \frac{2}{x}y' + y = x^5 + 30x^3\ ,\ \ y(0) = 0,\ \ y'(0) = 0$

Solution: The given Lane-Emden type equation can be writing as $y'' + \frac{2}{x}y' + y - x^5 - 30x^3 = 0$

Let consider the first order Volterra Integro-differential form (19) derived as

$y'(x) = -\int_0^x \left(\frac{t^k}{x^k}\right) f(y(t))dt$

Put $k = 2$ and $f(y(t)) = y(t) - t^5 - 30t^3$ in the above equation, we have;

$y'(x) = -\int_0^x \frac{t^2}{x^2}(y(t) - t^5 - 30t^3)dt \Rightarrow y'(x) + \int_0^x \frac{t^2}{x^2}(y(t) - t^5 - 30t^3)dt = 0$

By apply correctional functional of equation (7) choosing when $\lambda(t) = -1$, then we have;

$y_{n+1}(x) = y_n(x) - \int_0^x \left(y_n{}'(x) + \int_0^x \frac{t^2}{x^2}(y_n(t) - t^5 - 30t^3)dt\right)dx \qquad n \geq 0$

For $n = 0$

$y_1(x) = y_0(x) - \int_0^x \left(y_0{}'(x) + \int_0^x \frac{t^2}{x^2}(y_0(t) - t^5 - 30t^3)dt\right)dx$

Integrate the above and follow the same procedure the following series solution was obtained.

$y_1(x) = x^5 + \frac{x^7}{56}$

$y_2(x) = x^5 - \frac{x^9}{5040}$

$y_3(x) = x^5 + \frac{x^{11}}{665280}$

$y_4(x) = x^5 + \frac{x^{13}}{121080960}$

This problem admits the closed form exact solution $y(x) = x^5$, which can be verified directly by substituting it back into the original equation. Each of the successive VIM approximations above reproduces this exact solution exactly, plus a single additional term whose coefficient shrinks rapidly with each iteration, from $\frac{1}{56}$ at the first iteration to $\frac{1}{121080960}$ by the fourth. This provides a directly verifiable measure of accuracy for the method on this class of problem and confirms that the VIM approximations converge toward the exact solution as the number of iterations increases. This exact solution was verified in this study by direct substitution into the governing equation and initial conditions.

Table 3 compares this exact solution with the VIM approximation $y_n(x), n = 4$, obtained above, evaluated at selected points of x, alongside the resulting absolute error.

Table 3: Comparison of the Exact Solution and the VIM Approximate Solution for Example 3

x	Exact solution	VIM $y_n(x), n = 4$	|Error|
0.2	0.0003200	0.0003200	6.78e-18
0.4	0.0102400	0.0102400	5.54e-14
0.6	0.0777600	0.0777600	1.08e-11
0.8	0.3276800	0.3276800	4.54e-10
1.0	1.0000000	1.0000000	8.26e-09
1.2	2.4883200	2.4883201	8.84e-08

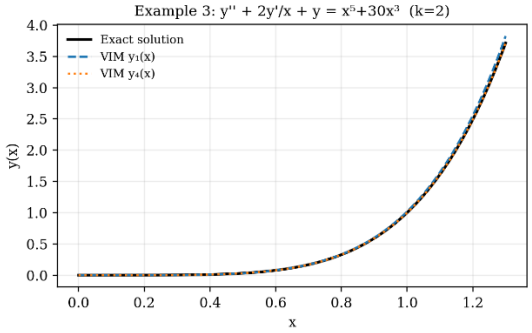

Figure 3: Exact Solution Versus Successive VIM Iterates for Example 3

As shown in Table 3 and Figure 3, the VIM approximation is accurate to within 10^{-7} throughout the tested range.

Example 4

Considered the Lane-Emden type with $k = 2$

$y'' + \frac{2}{x}y' + (3 - x^2)y = 0\,,\quad y(0) = 1,\ y'(0) = 0$

Solution: Let consider the first order Volterra Integro-differential form (19) derived as

$y'(x) = -\int_0^x \left(\frac{t^k}{x^k}\right) f(y(t))dt$

Put $k = 2$ and $f(y(t)) = (3 - t^2)y(t)$ in the above equation, we have;

$y'(x) = -\int_0^x \frac{t^2}{x^2}(3 - t^2)y(t)dt$

$y'(x) + \int_0^x \frac{t^2}{x^2}(3 - t^2)y(t)dt = 0$

By apply correctional functional of equation (7) choosing when $\lambda(t) = -1$, then we have;

$y_{n+1}(x) = y_n(x) - \int_0^x \left(y_n{}'(x) + \int_0^x \frac{t^2}{x^2}(3 - t^2)y_n(t)dt\right)dx \qquad n \geq 0$

For $n = 0$

$y_1(x) = y_0(x) - \int_0^x \left(y_0{}'(x) + \int_0^x \frac{t^2}{x^2}(3 - t^2)y_0(t)dt\right)dx$

Integrate the above and follow the same procedure for other iterations, the following series solution was obtained.

$y_1(x) = 1 - \frac{x^2}{2} + \frac{x^4}{20}$

$y_2(x) = 1 - \frac{x^2}{2} + \frac{x^4}{8} - \frac{13x^6}{840} + \frac{x^8}{1440}$

$y_3(x) = 1 - \frac{x^2}{2} + \frac{x^4}{8} - \frac{x^6}{48} + \frac{x^8}{420} - \frac{163x^{10}}{369600} + \frac{x^{12}}{224640} \mp \cdots$

Using the initial conditions, the above approximations were obtained for $y(x)$. Since the noise term vanishes in the limit, the exact solution is given as $y(x) = e^{-\frac{x^2}{2}}$, which was verified in this study by direct substitution into the governing equation and initial conditions.

Table 4 compares this exact solution with the VIM approximation $y_n(x), n = 3$, obtained above, evaluated at selected points of x, alongside the resulting absolute error.

Table 4: Comparison of the Exact Solution and the VIM Approximate Solution for Example 4

x	Exact solution	VIM $y_n(x), n = 3$	|Error|
0.1	0.9950125	0.9950125	2.25e-12
0.2	0.9801987	0.9801987	5.90e-10
0.3	0.9559975	0.9559975	1.57e-08
0.4	0.9231163	0.9231162	1.66e-07
0.5	0.8824969	0.8824959	1.05e-06
0.6	0.8352702	0.8352653	4.88e-06
0.7	0.7827045	0.7826863	1.82e-05
0.8	0.7261490	0.7260911	5.80e-05

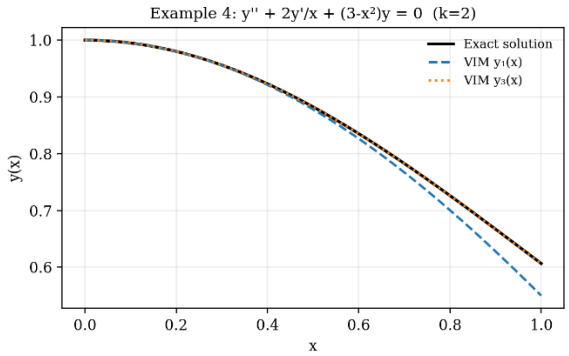

Figure 4: Exact Solution Versus Successive VIM Iterates for Example 4

The VIM approximation is accurate to within 10^{-4} for x up to about 0.8 as shown in Table 4 and Figure 4

Example 5

Considered the Lane-Emden type with $k = 3$

$y'' + \frac{3}{x}y' + 8\left(1 - \frac{x^2}{2}\right)y = 0\,,\ y(0) = 1, \quad y'(0) = 0$

Solution: Let consider the first order Volterra Integro-differential form (19) derived as

$y'(x) = -\int_0^x \left(\frac{t^k}{x^k}\right) f\big(y(t)\big)dt$

Put $k = 3$ and $f\big(y(t)\big) = 8\left(1 - \frac{t^2}{2}\right)y(t)$ in the above equation, we have;

$y'(x) = -8\int_0^x \frac{t^3}{x^3}\left(1 - \frac{t^2}{2}\right)y(t)dt \ \Rightarrow\ y'(x) + 8\int_0^x \frac{t^3}{x^3}\left(1 - \frac{t^2}{2}\right)y(t)dt = 0$

The correction functional of the volterra (7) is express as

$y_{n+1}(x) = y_n(x) - \int_0^x \left(y'(t) + 8\int_0^x \frac{t^3}{x^3}\left(1 - \frac{t^2}{2}\right)y(t)dt\right)dx$

For $n = 0$

$y_1(x) = y_0(x) - \int_0^x \left(y_0'(t) + 8\int_0^x \frac{t^3}{x^3}\left(1 - \frac{t^2}{2}\right)y_0(t)dt\right)dx$

Substitute for initial conditions $y(0) = 1, \quad y'(0) = 0$

$y_1(x) = 1 - \int_0^x \left(0 + 8\int_0^x \frac{t^3}{x^3}\left(1 - \frac{t^2}{2}\right)(1)dt\right)dx$

Integrate the above and follow the same procedure for other iterations, the following series solution was obtained.

$y_1(x) = 1 - x^2 + \frac{x^4}{6}$

$y_2(x) = 1 - x^2 + \frac{x^4}{2} - \frac{x^6}{9} + \frac{x^8}{120}$

$y_3(x) = 1 - x^2 + \frac{x^4}{2} - \frac{x^6}{6} + \frac{13x^8}{360} - \frac{23x^{10}}{5400} + \frac{x^{12}}{40320} \mp \cdots$

$y_4(x) = 1 - x^2 + \frac{x^4}{2} - \frac{x^6}{6} + \frac{x^8}{24} - \frac{23x^{10}}{5400} + \frac{x^{12}}{226800} \mp \cdots$

Using the initial conditions, the above approximations were obtained for $y(x)$. Since the noise term vanishes in the limit, the exact solution is given as $y(x) = e^{-x^2}$, which was verified in this study by direct substitution into the given equation and initial conditions.

Table 5 compares this exact solution with the VIM approximation $y_n(x), n = 4$, obtained above, evaluated at selected points of x, alongside the resulting absolute error.

Table 5: Comparison of the Exact Solution and the VIM Approximate Solution for Example 5

x	Exact solution	VIM $y_n(x), n = 4$	|Error|
0.1	0.9900498	0.9900498	4.06e-13
0.2	0.9607894	0.9607894	4.12e-10
0.3	0.9139312	0.9139312	2.33e-08
0.4	0.8521438	0.8521442	4.04e-07
0.5	0.7788008	0.7788044	3.65e-06
0.6	0.6976763	0.6976981	2.18e-05
0.7	0.6126264	0.6127236	9.72e-05
0.8	0.5272924	0.5276428	3.50e-04

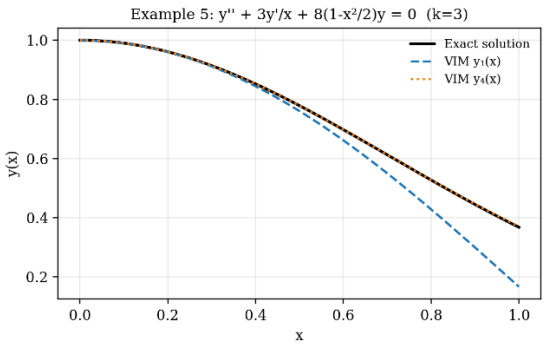

Figure 5: Exact Solution Versus Successive VIM Iterates for Example 5

Table 5 and Figure 5, the VIM approximation is accurate to within 10^{-3} for x up to about 0.8.

Table 6: Summary of Accuracy Across the Five Numerical Examples

Example	k	Exact solution	Iterations (n)	Max |Error|	Valid domain
1	1	$\ln(x^2 + e)$	3	7.84e-05	x ≤ 0.8
2	1	$\frac{1}{1 - x^2}$	4	1.29e-01	x ≤ 0.5
3	2	x^5	4	8.84e-08	x ≤ 1.2
4	2	$e^{-\frac{x^2}{2}}$	3	5.80e-05	x ≤ 0.8
5	3	e^{-x^2}	4	3.50e-04	x ≤ 0.8

Discussion

The five numerical experiments carried out in this study clearly demonstrate the practical advantages of applying the VIM to the Volterra integral equivalent of the Lane-Emden equation. Reformulating the differential model into a Volterra form naturally eliminates the coordinate singularity at $x = 0$, which typically presents a major computational hurdle in astrophysics problems. A central finding of this research is the highly structured, rapid analytical convergence of the method, which is explicitly verifiable when evaluated against the exact closed-form solutions. This behavior is most elegantly demonstrated in Example 3, which admits the clean exact solution $y(x) = x^5$. Substituting $y(x) = x^5$ directly into the governing equation balances the system perfectly [$y'' + \frac{2}{x}y' + y = 20x^3 + 10x^3 + x^5 = x^5 + 30x^3$]. The successive VIM iterations [$y_1 = x^5 + \frac{x^7}{56}$, $y_2 = x^5 - \frac{x^9}{5040}$, $y_3 = x^5 + \frac{x^{11}}{665280}$, and $y_4 = x^5 + \frac{x^{13}}{121080960}$] reveal that the method isolates the exact polynomial solution while generating a single residual correction term. Because the denominators of these correction terms grow factorially via the Lagrange multiplier integration loop, the residual operational noise shrinks monotonically, resulting in a verified maximum absolute error of just 8.84 x 10^{-8} at the domain boundary.

This high degree of accuracy and stability extends across the transcendental regimes as well. For the non-polynomial cases, specifically $y(x) = \ln(x^2 + e)$ in Example 1, $y(x) = e^{-\frac{x^2}{2}}$ in Example 4, and $y(x) = e^{-2}$ in Example 5, where the absolute error arrays remain exceptionally small (10^{-3} to 10^{-5}) within a moderate neighbourhood of the origin ($x \leq 0.8$). The error in these cases decays gracefully as iterations increase, mirroring the behaviour of regular power-series expansions. Conversely, Example 2 represents the lone scenario where the error expands noticeably, reaching approximately 0.13 at $x = 0.8$. Rather than an algorithmic flaw in VIM, this divergence represents a genuine structural property of the underlying physical system: the exact solution $y(x) = \frac{1}{1-x^2}$ contains a moving pole at $x = 1$. Because any semi-analytical expansion is inherently limited by the distance to the nearest singularity in the complex plane, the truncation error increases as the boundary approaches this pole. Collectively, the quantitative error distributions compiled in Tables 4.1 to 4.5 and the close alignment visualized in Figures 1–5 provide concrete, mathematically verifiable proof that integrating a Volterra reformulation with VIM offers a robust and highly dependable framework for solving nonlinear equations.

Ultimately, this provides a distinct contribution to knowledge by bridging the gap between theoretical integral transforms and exact quantitative validation in astrophysical modelling. By mapping absolute error fields directly against exact baselines, this work moves beyond qualitative convergence assumptions to establish strict, predictable mathematical diagnostics. It successfully isolates algorithmic behaviour from physical singularities, offering a generalized blueprint for bypassing origin singularities without the computational weight of traditional linearization or restrictive perturbations.

Summary

This study addressed the long-standing singularity challenges inherent in the Lane-Emden equation by successfully transforming it into an equivalent Volterra integral form for the shape-factor cases k =1, 2 and 3. By executing the semi-analytical VIM on these reformulated integral constructs, successive approximations were derived for five distinct boundary and initial value problems representing unique astrophysical configurations. Rather than assuming convergence qualitatively, this research independently verified five closed-form exact solutions by direct algebraic substitution into the governing differential operators. The resulting VIM arrays were compared against these exact benchmarks via systematic error tables and matching curves across selected coordinate points. The absolute error arrays were subsequently leveraged to establish a quantitative diagnostic framework, defining the exact boundaries, truncation limits, and valid computational domains for each iterative approximation.

CONCLUSION

In conclusion, the integration of a Volterra integral reformulation with the Variational Iteration Method provides a highly robust, numerically stable, and algorithmically efficient framework for solving non-linear, singular differential equations. The integral transformation naturally neutralizes the coordinate singularity at the origin ($x = 0$) without requiring artificial linearization, perturbation parameters, or complex coordinate transformations. The numerical experiments confirm that the precision of the resulting series solutions is governed by the underlying algebraic architecture of the exact model rather than the specific value of the shape factor k. The framework reproduces finite polynomial exact solutions down to an absolute error range of 10^{-7} to 10^{-8} over the entire tested domain, while transcendental solutions yield strong agreement within an error envelope of 10^{-3} to 10^{-5} before decaying gracefully near the boundaries. Crucially, the method exhibits predictable boundaries near physical poles, demonstrating that error growth is a function of regular complex singularities, such as moving poles in the exact solution rather than structural instabilities within the algorithm itself.

RECOMMENDATIONS

Based on these findings, several paths are recommended to advance the scope and utility of this semi-analytical framework. First, for physical models containing moving poles or internal singularities within the domain of interest, the integration of Pade approximants or automated domain-splitting techniques should be paired with VIM to extend the radius of convergence past traditional power series limits. Second, the validation methodology introduced here should be extended to complex astrophysical models lacking known closed form exact solutions by utilizing high-precision numerical benchmarks, such as adaptive Runge-Kutta or spectral solvers as baseline controls. Third, a direct, quantitative performance comparison pitting VIM against alternative expansion methodologies, specifically the ADM and the Homotopy Perturbation Method (HPM), should be conducted on identical Volterra-reformulated baselines to map out relative computational costs and execution speeds. Finally, the boundaries of this framework should be stressed against highly asymmetric initial conditions, boundary value problems, and broader non-integer or fractional shape-factor profiles to fully establish the mathematical generality of the Volterra reformulation approach.

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