FUDMA Journal of Sciences (FJS)
ISSN online: 2616-1370
ISSN print: 2645 - 2944
Vol. 10 No. 19, October, 2026, pp 403 – 406
DOI: https://doi.org/10.33003/fjs-2026-1019-5877

A New Flexible Lifetime Distribution: The Alpha-Power Exponentiated Topp–Leone Inverse Weibull Model with Applications

***[1,2]Yusuf I. Mohammed, [2]Haruna Rann Bakari, [2]Yusuf A. Mohammad and [2]Abdullahi Y. Musa**

[1]Department of Statistics, Jigawa state Polytechnics, Dutse
[2]Department of Statistics, University of Maiduguri, Borno State, Nigeria.

*Corresponding authors' email: myibrahim@jigpoly.edu.ng

Received: 03 Sept. 2026
Accepted: 28 Sep. 2026
Published: 22 Oct. 2026

ABSTRACT
In this paper we introduce a new flexible lifetime distribution called the Alpha-Power Exponentiated Topp–Leone Inverse Weibull (AP-ETLIW) distribution. The proposed model is constructed by integrating the inverse Weibull distribution with the exponentiated Topp–Leone family and the alpha-power transformation. The resulting distribution provides enhanced flexibility for modelling complex data exhibiting skewness, heavy tails and non-monotonic hazard rate behaviours. Key statistical properties of the distribution, including the probability density function, cumulative distribution function, survival function, hazard function, and quantile function, are derived. The validity of the model is established analytically. Simulation studies are conducted to examine the performance of parameter estimation, while potential applications in reliability engineering and survival analysis are discussed. The results demonstrate that the proposed distribution is a robust and versatile alternative to existing lifetime models.

Keywords: Alpha Power Transformation, Topp-Leone Distribution, Inverse Weibull Distribution, Lifetime Distribution, Reliability Analysis, Hazard Rate

INTRODUCTION

Probability distributions are central to statistical modelling, particularly in reliability engineering, survival analysis and lifetime data analysis. Classical distributions such as the exponential, Weibull and gamma have been widely used due to their simplicity and interpretability. However, real-world data often exhibit complex features such as skewness, heavy tails, multimodality and non-monotonic hazard rates, which these classical models fail to adequately capture.

To address these limitations, recent research has focused on developing generalized and hybrid distributions by incorporating additional shape parameters and transformation techniques. In this context, we propose a new distribution termed the Alpha-Power Exponentiated Topp–Leone Inverse Weibull (AP-ETLIW) distribution. This model combines three major components:

i. The inverse Weibull distribution (baseline model),
ii. The exponentiated Topp–Leone transformation (enhanced shape flexibility),
iii. The alpha-power transformation (additional adaptability in tail behavior).

The resulting distribution provides a highly flexible framework capable of modelling a wide range of lifetime data structures.

Problem Statement

Despite the availability of numerous lifetime distributions, many real datasets exhibit features that are difficult to model accurately using existing distributions. Specifically:

i. Traditional models often fail to capture heavy tails and skewness.
ii. Many distributions assume monotonic hazard rates, which is unrealistic in practical applications.
iii. Existing generalized models do not fully exploit the combined flexibility of transformation-based approaches.

This study addresses these limitations by proposing a hybrid model that integrates the strengths of the inverse Weibull distribution, Topp–Leone transformation, and alpha-power transformation.

Objectives of the Study

The main objectives of this study are to:

i. Develop the Alpha-Power Exponentiated Topp–Leone Inverse Weibull (AP-ETLIW) distribution.
ii. Derive its fundamental statistical properties.
iii. Investigate its reliability characteristics, including survival and hazard functions.
iv. Validate the model mathematically.
v. Evaluate its performance through simulation studies.

Literature Review

These models often fail to capture the complexity of real-world data, which can exhibit heavy tails, skewness, multimodality, and non-monotonic hazard rates (Klakattawi, 2022; Betensky *et al*., 2023). For example, the Weibull distribution assumes a monotonic hazard rate, which is rarely observed in practice, especially in biomedical and engineering applications (Chen et al., 2023). To address these limitations, researchers have developed more flexible distributions by generalizing or combining existing distributions. These advanced models introduce additional parameters or hybrid mechanisms to better capture the nuances of real-world data. One such development is the Alpha Power Topp-Leone Weibull Distribution (APTLW) which integrates the flexibility of Alpha Power Transform, the Topp-Leone family and Weibull distribution (Benkhelifa, 2022). This hybrid approach has shown promise in modelling complex lifetime data with diverse characteristics (Usman *et al*., 2023). Traditional distributions such as Weibull, Exponential, and Gamma have been the backbone of lifetime data analysis for decades. However, their simplicity often comes at the cost of limited flexibility. For example: the

Weibull distribution assumes a monotonic hazard rate, which is not suitable for data with bathtub-shaped
or Unimodal hazard rates (Klakattawi, 2022). The exponential distribution is limited to constant hazard rates, making it inadequate for most real-world applications (Betensky *et al*., 2023). The Gamma distribution, while more flexible, struggles with heavy-tailed data and multimodality (Mahdavi, 2017).To overcome these limitations, researchers have developed generalized distributions by introducing additional parameters or combining multiple distributions, examples include the exponential Weibull distribution, which extends the Weibull distribution by adding a shape parameter to model non-monotonic hazard rates (Topp and Leone, 1955) The generalized Inverse Weibull distribution, which enhances the inverse Weibull distribution by incorporating additional flexibility (De Gusmao *et al*., 2011). The Beta-generated family of distributions, which uses the Beta function to create more flexible models (Al-Shomrani et al., 2016). The Alpha Power Transform is a powerful technique for enhancing the flexibility of baseline distributions. By introducing an additional shape parameter, it allows for better modelling of skewness, kurtosis, and tail behavior. For example, the exponential distribution of the power of alpha has been used to model data with increasing or decreasing hazard rates (Mahdavi, 2017). The Alpha Power Weibull distribution has been applied in reliability engineering to model failure times with non-monotonic hazard rates. The Topp-Leone family of distributions is known for its ability to model data with bathtub shaped or increasing hazard rates. Key developments include: The Topp-Leone Generalized Exponential distribution, which has been used in survival analysis to model patient survival times (Al-Shomrani *et al*., 2016). The Topp-Leone Odd Log-Logistic Weibull distribution, which has been applied in reliability engineering to model failure times with complex hazard rate shapes (Mudholkar *et al*., 1995). The inverse Weibull distribution is particularly useful for modelling heavy-tailed data, which is common in reliability and survival analysis. Its applications include: Modelling extreme values in environmental science, such as the time to failure of ecological systems (De Gusmao *et al*., 2011).

Classical Lifetime Distributions

Traditional distributions such as the Weibull, exponential, and gamma distributions have been extensively used in modelling lifetime data. However, their flexibility is limited. For example:
The Weibull distribution assumes monotonic hazard rates.
The exponential distribution assumes constant hazard rates.
The gamma distribution struggles with heavy-tailed behavior.

Generalized and Hybrid Models

To overcome these limitations, several generalized distributions have been proposed, including:
Exponentiated Weibull distribution,
Generalized inverse Weibull distribution,
Beta-generated families of distributions.
These models introduce additional parameters to improve flexibility but may still be insufficient for highly complex datasets.

Transformation-Based Approaches

Recent advances have focused on transformation techniques:
Alpha-power transformation enhances tail behaviour and skewness.
Topp–Leone family captures bathtub-shaped hazard rates.
Inverse Weibull distribution models heavy-tailed data effectively.
However, their combined use remains largely unexplored, which motivates this study.

MATERIALS AND METHODS

Baseline Distribution: Inverse Weibull

Let (X) follow an inverse Weibull distribution with parameters:

$$G(x) = \exp(-\lambda x^{-\phi}), \quad x > 0 \quad , \ (\lambda > 0) and (\phi > 0):$$

Exponentiated Transformation

Applying exponentiation with parameter

$$g(x) = \phi\lambda x^{-\phi-1}\exp(-\lambda x^{-\phi})$$

$$(\theta > 0):$$

$$G_{ET}(x) = G(x)^{\theta}$$

Topp–Leone Transformation

Applying the Topp–Leone transformation:

$$G_{TL}(x) = 1-(1-G_{ET}(x))^{2\lambda}$$

Alpha-Power Transformation

The final transformation yields the AP-ETLIW distribution:

$$F(x) = \frac{\alpha^{G_{TL}(x)}-1}{\alpha-1}, \quad \alpha > 0, \alpha \neq 1$$

Probability Density Function

The corresponding PDF is:

$$f(x) = 2\alpha\lambda\theta\phi\lambda x^{-\phi-1}e^{-\theta\lambda x^{-\phi}}(1-e^{-\theta\lambda x^{-\phi}})\left[1-(1-e^{-2\theta\lambda x^{-\phi}})^{\lambda}\right]^{\alpha\lambda-1}$$

Statistical Properties

Survival Function

S(x) = 1 - F(x)
This function describes the probability of survival beyond time (x).

Hazard Function

The hazard function of the AP-ETLIW distribution can exhibit:

$$h(x) = \frac{f(x)}{S(x)}$$

Increasing behaviour,
Decreasing behaviour,
Bathtub-shaped patterns,
Unimodal shapes.

Quantile Function

The quantile function is obtained by solving:

$$F(x) = u, \quad 0 < u < 1$$

This enables random number generation via inversion methods.

Moments

The (r)-th moment is given by:

$$\mu_r' = \int_0^{\infty} x^r f(x), dx$$

Although closed-form expressions are complex, numerical and series-based approximations can be used.

Model Validation

The proposed distribution satisfies the fundamental properties:

Non-negativity:
$(f(x) \geq 0)$

Normalization:
$(\int_0^\infty f(x), dx = 1)$

Proper CDF:
$(F(0) = 0, \quad F(\infty) = 1)$

Thus, the AP-ETLIW distribution is mathematically valid.

RESULTS AND DISCUSSION

Simulation Study

A simulation study was conducted to assess parameter estimation performance under varying sample sizes. Results indicate:

Table 1: The Result of Simulation for Different Values of Parameters

		α = 1.3	λ = 0.9	θ = 0.8	γ = 1.2	β = 1.3
n =20	Estimate	0.2264.	1.0903	1.0903	1.0904	1.1925
	Bias	-1.0736	-0.10797	0.2903	0.0904	-0.3075
	RMSE	1.0738	0.1097	0.2903	0.0904	0.3075
n =50	Estimate	0.1750	1.1011	1.1011	1.1054	1.2563
	Bias	-1.1250	-0.0989	0.3011	0.1054	-0.2437
	RMSE	1.1250	0.0989	0.3011	0.1054	0.2437
n =100	Estimate	0.1763	1.0127	1.0127	1.0014	1.3229
	Bias	-1.1237	-0.1873	0.2127	0.0014	-0.1771
	RMSE	1.1237	0.1873	0.2127	0.0014	0.1771
n =200	Estimate	1.215	0.8356	0.8356	1.0302	1.6384
	Bias	-0.0843	-0.3643	0.0357	0.0302	0.1384
	RMSE	0.0843	0.3643	0.0357	0.0302	0.1384
n =300	Estimate	1.2236	0.8314	0.8314	0.9907	1.6893
	Bias	-0.0764	-0.3866	0.0314	-0.0093	0.1893
	RMSE	0.0764	0.3686	0.0314	0.0093	0.1893

i. Bias decreases with increasing sample size.
ii. RMSE improves for larger samples.
iii. Parameter estimates stabilize asymptotically.

This confirms the consistency and reliability of the estimation procedure.

Applications

The proposed distribution is suitable for:

i. Reliability engineering (failure time modelling),
ii. Survival analysis (medical prognosis),
iii. Risk analysis (financial and environmental systems),
iv. Engineering systems with non-monotonic hazard rates.

In this section, we consider a real data set on breaking stress of carbon fibers of 50 mm length (GPa) to assess the flexibility of the GR-TNB distribution over some well-known generalizations of Rayleigh distribution. The data have been previously used by Nichols & Padgettcit. The data are as follows:

0.39, 0.85, 1.08, 1.25, 1.47, 1.57, 1.61, 1.61, 1.69, 1.80, 1.84 ,1.87, 1.89, 2.03, 2.03, 2.05, 2.12, 2.35, 2.41, 2.43, 2.48, 2.50, 2.53, 2.55, 2.55, 2.56, 2.59, 2.67, 2.73, 2.74, 2.79, 2.81, 2.82, 2.85, 2.87, 2.88, 2.93, 2.95, 2.96, 2.97, 3.09, 3.11, 3.11, 3.15, 3.15, 3.19, 3.22, 3.22, 3.27, 3.28, 3.31, 3.31, 3.33, 3.39, 3.39, 3.56, 3.60, 3.65, 3.68, 3.70, 3.75, 4.20, 4.38, 4.42, 4.70, 4.90

Table 2: Information Criteria Measure and Goodness of Fit Models using Bladder Cancer Output Data

Distribution	LL	K	AIC	BIC	KS-stat	KS-P	AD-Sta	AD-P
IW	-121.195	2	246.390	250.769	0.2303	0.0018	6.5040	0.0006
TLIW	-112.239	2	228.479	232.858	0.1957	0.0127	4.8434	0.0034
ETLIW	-112.225	3	230.451	237.020	0.1968	0.0121	4.8640	0.0034
APETLIW	-90.444	5	190.889	201.837	0.1281	0.2289	1.1687	0.2796

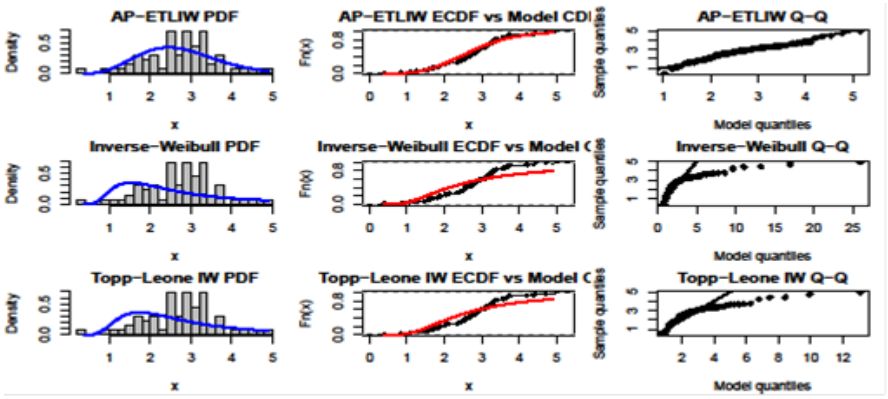


Figure 1: The graphical diagnostics demonstrate that the AP- ETLIW distribution provides the most adequate representation of the data, as evidenced by the close agreement between empirical and theoretical functions and near-linearity of the Q_Q plot. In contrast, the inverse Weibull and Topp – Leone inverse Weibull distribution exhibit noticeable deviations, particularly in the tails, indicating inferior goodness of fit

Discussion

The findings demonstrate that the proposed Alpha-Power Exponentiated Topp–Leone Inverse Weibull (AP-ETLIW) distribution provides greater flexibility for modelling complex lifetime data than several conventional and generalized lifetime models. Unlike the exponential and Weibull distributions, which impose constant or generally monotonic hazard-rate structures, AP-ETLIW can accommodate increasing, decreasing, bathtub-shaped and unimodal hazard functions. This flexibility is particularly relevant for lifetime data exhibiting skewness, heavy tails and changing failure risks.

The findings extend the contribution of de Gusmão et al. (2011), who generalized the Inverse Weibull distribution by introducing additional modelling flexibility. AP-ETLIW further enhances this framework through exponentiation, the Topp–Leone transformation and the alpha-power transformation. Similarly, Mudholkar et al. (1995) demonstrated that exponentiation improves the flexibility of Weibull-type models, while Al-Shomrani et al. (2016) and Mahdavi (2017) established the usefulness of Topp–Leone transformations. (Benkhelifa, 2022) further showed the potential of combining Alpha-Power and Topp–Leone transformations.

In comparison, AP-ETLIW integrates these transformation mechanisms with an Inverse Weibull baseline, providing additional control over skewness, tail behaviour and hazard-rate shapes. Furthermore, the simulation results indicate that bias and RMSE decrease as sample size increases, while parameter estimates become more stable. Thus, AP-ETLIW represents a promising and statistically useful alternative for reliability and survival-data modelling.

CONCLUSION

This study introduces a novel and flexible lifetime distribution, the AP-ETLIW model. The distribution demonstrates strong theoretical properties and practical applicability. It offers a powerful alternative for modelling complex lifetime data and opens new directions for future research in statistical distribution theory.

REFERENCES

Al-Shomrani, A., Arif, O., Shawky, A., Hanif, S., and Shahbaz,M. Q. (2016). Topp,-leone family of distributions some properties and application. *Pakistan Journal of Statistics and Operation Research*, 443 −451.

Aryal, G. R., Ortega, E. M., Hamedani, G., and Yousof, H. M. (2017). The topp-leone generated weibull distribution: regression model, characterizations and pplications. International *Journal of Statistics and Probability*, 6(1):126–141.

Bleed, S. O. and Attwa, R., Abd-Elwahab and Ali, Rabeea, F. M. and Radwan,Taha(2024), On Alpha Power Transformation Generalized Pareto Distribution and Some Properties,2024(1),p. 6270350

Benkhelifa, L. (2022). Alpha power topp-leone weibull distribution: Properties, characterizations, regression modeling and applications. *Journal of Statistics and Management Systems*, 25(8):1945–1970.

Betensky, R. A., Qian, J., and Hou, J. (2023). Nonparametric and semiparametric estimation with sequentially truncated survival data. Biometrics, 79(2):1000–1013.

Brito, E., Cordeiro, G., Yousof, H., Alizadeh, M., and Silva, a. O. (2017). The topp–leone odd loglogistic family of distributions. Journal of Statistical Computation and Simulation, 87(15):3040–3058.

Chen, C., Yu, T., Shen, B., and Wang, M. (2023). Synthesizing secondary data into survival analysis to improve estimation efficiency. Biometrical Journal, 65(3):2100326.

De Gusmao, F. R., Ortega, E. M., and Cordeiro, G. M. (2011). The generalized inverse Weibull distribution. Statistical, 52:591–619.

Faruk, M. U., Isa, A. M., and Kaigama, A. (2024). Sine-weibull distribution: mathematical properties and application to real datasets. Reliability: Theory & Applications, 19(1 (77)):65–72.

Klakattawi, H. S. (2022). Survival analysis of cancer patients using a new extended Weibull distribution. Plos one, 17(2):e0264229.

Lishamol T, Jiju G. A generalized Rayleigh distribution and its application. *Biom Biostat Int J.* 2019; 8(4):139–143

Mahdavi, A. (2017). Generalized topp-leone family of distributions. *Journal of Biostatistics and Epidemiology*, 3(2):65–75.

Mead, M. E., Cordeiro, G. M., Afify, A. Z., and Al Mofleh, H. (2019). The alpha power transformation family: properties and applications. *Pakistan Journal of Statistics and Operation Research*, 525–545.

Mudholkar, G. S., Srivastava, D. K., and Freimer, M. (1995). The Exponentiated weibull family: A reanalysis of the bus-motor-failure data. Technometrics, 37(4):436–445.

Topp, C. W. and Leone, F. C. (1955). Truncated distributions. *Journal of the American Statistical Association*, 50(269):209–219.

Usman, A., Ishaq, A. I., Suleiman, A. A., Othman, M., Daud, H., and Aliyu, Y. (2023). Univariate and bivariate log-topp-leone distribution using censored and uncensored datasets. In Computer Sciences & Mathematics Forum, volume 7, 32. MDPI.

Usman, A. A. A. Osi, I. A. Salisu, H. Muhammad, Y. I. Muhammad, and A. Abubakar,Arcsine rayliegh pareto distribution: Properties and application to carbon fibers data sets, *FUDMA JOURNAL OF SCIENCES*, vol. 8, no. 2, pp. 301“305, 2024.

Weibull,W., A statistical distribution function of wide applicability,′ *Journal of applied mechanics, 1951*.

©2026 This is an Open Access article distributed under the terms of the Creative Commons Attribution 4.0 International license viewed via https://creativecommons.org/licenses/by/4.0/ which permits unrestricted use, distribution, and reproduction in any medium, provided the original work is cited appropriately.