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Eigenvalue Asymptotic for Sturm–Liouville Problems with Nonlocal Two-Point Boundary Conditions

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ABSTRACT
This paper investigates a one-dimensional Sturm–Liouville problem with a local Dirichlet and a nonlocal two-point boundary condition. Assuming the potential function $q \in C^r[0,1]$, we construct the fundamental solution and derive precise asymptotic expansions for the characteristic function. This allows us to localize the characteristic roots and obtain first-order asymptotic formulas for the spectral parameters. Our main result proves that the eigenvalues follow a Weyl-type law $\lambda_k \sim \pi^2\left(k-\frac{1}{2}\right)^2$, $k \to \infty$, demonstrating that nonlocal conditions modify only lower-order spectral corrections while preserving classical quadratic growth.

Keywords: Sturm–Liouville Problem, Nonlocal Boundary Conditions, Characteristic Function, Weyl-Type Law, Eigenvalue Asymptotic, Spectral Analysis

INTRODUCTION

Sturm–Liouville problems are fundamental in spectral theory and mathematical physics (Levitan & Sargsjan, 1991). While classical boundary conditions are well-understood, nonlocal boundary conditions—where the solution's behavior at one endpoint depends on an interior point—have gained significant attention for modeling complex physical phenomena and discrete systems (Liu et al., 2024; Zhang, 2026). Recent literature has extensively analyzed asymptotic eigenvalue distributions for various nonlocal boundary types, including integral (Štikonas & Şen, 2021a, 2021b), Neumann (Štikonas & Şen , 2022), and Robin conditions (Štikonas, 2022). This growing interest also extends to non-real eigenvalue bounds (Sun et al., 2022; Sun & Li, 2023), inverse spectral and nodal problems (Özkan & Adalar, 2023), and nonlinear differential inclusions (Alama, 2025).

In this paper, we consider the one-dimensional Sturm–Liouville problem

$$-u''(t) + q(t)u(t) = \lambda u(t),\ 0 < t < 1, \qquad (1)$$

With the boundary conditions

$$u(0) = 0,\ u'(1) = \gamma u(\xi) \qquad (2)$$

Where q is a real-valued potential function, ($\gamma \in R$) and $\xi \in [0,1]$

The main objective is to investigate the asymptotic behavior of the characteristic function and its zeros. Under the regularity assumption

$q \in C^r[0,1])\ , r \in N_0$,

While the foundational studies by (Štikonas & Şen, 2021a, 2023) have significantly advanced the spectral analysis of nonlocal Sturm–Liouville operators, their primary focus centered on integral-type boundary conditions (Štikonas, 2022; Štikonas & Şen, 2021a, 2021b, 2022). This paper tackles a structurally distinct configuration: a local Dirichlet condition at one endpoint coupled with a nonlocal derivative condition, $u'(1) = \gamma u(\xi)$. Furthermore, unlike prior works that often rely on minimal $L^1(0,1)$ integrability to find basic eigenvalue distributions, we impose a stronger regularity hypothesis, $q \in C^r[0,1]$ (with $r \geq 1$). This specific assumption enables us to advance the Volterra integral technique beyond leading-order estimates to derive explicit, arbitrary-order asymptotic expansions. Crucially, our proof technique precisely isolates the first-order spectral corrections, mathematically decoupling the contribution of the integrated potential from the interference generated by the nonlocal parameters. Therefore, the standalone aim of this paper is to rigorously derive the exact eigenvalue asymptotics and establish the precise Weyl-type law for this specific class of non-self-adjoint boundary value problems.

Mathematical Preliminaries and Problem Formulation

Sturm–Liouville Problem

We consider the following Sturm–Liouville problem on the finite interval [0,1]:

$$-u''(t) + q(t)u(t) = \lambda u(t),\ 0 < t < 1, \qquad (3)$$

subject to the nonlocal boundary conditions

$$u(0) = 0,\ u'(1) = \gamma u(\xi) \qquad (4)$$

where $\lambda \in C$ is the spectral parameter, q is a real-valued potential function, $\gamma \in R$, and $\xi \in (0,1)$ (Štikonas & Şen, 2023)is a fixed interior point.

The presence of the term $u(\xi)$ in the second boundary condition makes the problem nonlocal, since the boundary value at $t = 1$ depends on the value of the solution at an interior point of the interval. Such boundary conditions constitute an important class of nonlocal Sturm–Liouville problems and have been investigated in several studies concerning characteristic functions and spectral asymptotics.

Throughout the analysis, we assume that

$q \in C^r[0,1])\ , r \in N_0$,

Whenever higher-order asymptotic expansions are considered. For the first-order asymptotic analysis, weaker assumptions on q, such as $q \in L1(0,1)$, may be sufficient.

Introducing the spectral parameter

$$\lambda = s^2, \qquad (5)$$

The differential equation can be written in the form

$$-u''(t) + q(t)u(t) = s^2 u(t), \qquad (6)$$

Or equivalently,

$$u''(t) + (s^2 - q(t))u(t) = 0. \qquad (7)$$

This representation is particularly convenient for studying the behavior of the solutions as $|\ s\ | \to \infty$, (Eastham, 1989; Levitan & Sargsjan, 1991)

To formulate the spectral problem in a rigorous operator-theoretic framework, we define the associated differential

operator L acting on the Hilbert space $L^2(0,1)$. The operator is given by the differential expression:

$$Lu = -u'' + q(t)u \quad (8)$$

with its domain $D(L)$ strictly defined to incorporate the nonlocal boundary conditions:

$$D(L) = \{u \in L^2\text{^}(0,1) : u, u' \text{ are absolutely continuous}\},$$
$$u'' \in L^2(0,1), u(0) = 0, u'(1) = \gamma\, u(\xi)\} \quad (9)$$

A critical consequence of the interior evaluation term $u(\xi)$ in the boundary condition is that the operator L is inherently **non-**self-adjoint ($L \neq L^*$). Using standard integration by parts for any $u, v \in D(L)$, the boundary terms do not vanish symmetrically. This non-self-adjoint nature has profound spectral implications: the classical spectral theorem is inapplicable, the eigenvalues are not guaranteed to be strictly real, and the corresponding eigenfunctions may not form a standard orthogonal basis. Consequently, analyzing the roots of the characteristic function through rigorous asymptotic expansions becomes the primary and most reliable method to extract the eigenvalue distribution for this non-self-adjoint nonlocal problem.

Fundamental Solution

To construct the characteristic function of the problem, we introduce a fundamental solution $\omega(t,s)$ of the differential equation satisfying the initial conditions

$$\omega(0,s) = 0,\ \omega'(0,s) = -1. \quad (10)$$

Thus, $\omega(t,s)$ is defined as the solution of the initial-value problem

$$\omega''(t,s) + (s^2 - q(t))\omega(t,s) = 0, \quad (11)$$

With

$$\omega(0,s) = 0,\ \omega'(0,s) = -1. \quad (12)$$

The choice of the normalization $\omega'(0,s) = -1$ is convenient for obtaining a unique solution and does not affect the location of the characteristic roots.

When $q(t) = 0$, the corresponding unperturbed equation becomes

$$\omega''(t,s) + s^2\omega(t,s) = 0. \quad (13)$$

Using the initial conditions, its solution is

$$\omega_0(t,s) = \frac{-\sin(st)}{s}. \quad (14)$$

Consequently, for a general potential q, the function $\omega(t,s)$ can be regarded as a perturbation of the unperturbed solution $-\sin(st)/s$.

This observation forms the basis of the asymptotic analysis developed in the subsequent sections.

Integral Representation of the Fundamental Solution

The differential equation can be transformed into an equivalent Volterra integral equation. Applying the variation-of-parameters method to

$$\omega''(t,s) + s^2\omega(t,s) = q(t)\omega(t,s), \quad (15)$$

and using the initial conditions at $t = 0$, we obtain

$$\omega(t,s) = \frac{-\sin(st)}{s} + \frac{1}{s}\int_0^t \sin\big(s(t-\tau)\big)\, q(\tau)\omega(\tau,s)\, d\tau \quad (16)$$

for $0 \le t \le 1$.

This integral representation is fundamental for deriving asymptotic estimates of the solution. In particular, the first term represents the solution of the unperturbed equation, while the integral term contains the effect of the potential q.

For large values of $|s|$, the integral equation allows successive approximations to be constructed. Under appropriate assumptions on q, the first approximation is

$$\omega(t,s) = \frac{-\sin(st)}{s} + O\left(\frac{1}{s^2}\right),\ |\, s\,| \to \infty. \quad (17)$$

More refined substitutions into the integral equation lead to higher-order asymptotic expansions, which will be developed in Section 3.

Derivative of the Fundamental Solution

Differentiating the integral representation with respect to t, we obtain

$$\omega'(t,s) = -cos(st) + \int_0^t cos(s(t-\tau))q(\tau)\omega(\tau,s)\, d\tau. \quad (18)$$

Thus,

$$\left[\omega'(t,s) = -\cos(st) + \int_0^t \cos\big(s(t-\tau)\big)\, q(\tau)\omega(\tau,s)\, d\tau\right] \quad (19)$$

and, in particular, at $t = 1$,

$$\omega'(1,s) = -\cos\, s + \int_0^1 \cos\big(s(1-\tau)\big)\, q(\tau)\omega(\tau,s)\, d\tau$$

Similarly, evaluating the fundamental solution at the interior point $t = \xi$ gives

$$\omega(\xi,s) = \frac{-sin(\xi s)}{s} + \frac{1}{s}\int_0^{\xi} sin(s(\xi-\tau))q(\tau)\omega(\tau,s)\, d\tau.$$

These two expressions are the main components used to construct the characteristic function associated with the nonlocal boundary condition.

Characteristic Function

The second boundary condition is

$$u'(1) = \gamma u(\xi). \quad (20)$$

Since the first boundary condition $u(0) = 0$ is already incorporated into the definition of $\omega(t,s)$, a nontrivial solution of the original boundary-value problem exists precisely when

$$\omega'(1,s) - \gamma\omega(\xi,s) = 0. \quad (21)$$

Therefore, we define the characteristic function by

$$h(s) = \omega'(1,s) - \gamma\omega(\xi,s) \quad (22)$$

And the characteristic values are the zeros of h.

Substituting the integral representations obtained above yields

$$h(s) = -\cos s + \int_0^t \cos\big(s(1-\tau)\big)\, q(\tau)\omega(\tau,s)\, d\tau + \frac{\gamma\sin(\xi s)}{s} - \frac{\gamma}{s}\int_0^{\xi} sin(s(\xi-\tau))q(\tau)\omega(\tau,s)\, d\tau \quad (23)$$

Hence, the characteristic equation is

$$h(s) = 0. \quad (24)$$

The zeros of $h(s)$ (Štikonas & Şen, 2021b, 2023) determine the spectral parameters s_k, and the corresponding eigenvalues are given by

$$\lambda_k = s_k^2. \quad (25)$$

This formulation reduces the spectral problem to the analysis of the zeros of a single characteristic function.

Unperturbed Characteristic Function

It is useful to consider first the case

$$q(t) \equiv 0. \quad (26)$$

In this case,

$$\omega(t,s) = -\frac{sin(st)}{s}, \quad (27)$$

And therefore

$$\omega'(t,s) = -cos(st). \quad (28)$$

Consequently, the characteristic function becomes

$$h_0(s) = -cos\, s + \frac{\gamma sin(\xi s)}{s}. \quad (29)$$

Thus, the unperturbed characteristic equation is

$$-cos\, s + \frac{\gamma sin(\xi s)}{s} = 0. \quad (30)$$

For large |s|, the second term is of order $O(|\, s\,|^{-1})$, whereas the first term is of order one. Therefore, the dominant part of the characteristic equation is

$$cos\, s = 0. \quad (31)$$

The corresponding leading-order characteristic points are

$$x_k = \pi\left(k - \frac{1}{2}\right),\ k = 1,2,\ldots \quad (32)$$

Which satisfy

$$cos\, x_k = 0, \qquad sin\, x_k = (-1)^{k-1}. \quad (33)$$

These points provide the natural reference sequence around which the characteristic roots s_k of the perturbed problem are localized.

Preliminary Asymptotic Quantities

For the asymptotic analysis, it is convenient to introduce the integrated potential

$$Q(t) = \frac{1}{2}\int_0^t q(\tau)\,d\tau. \quad (34)$$

In particular,

$$Q(1) = \frac{1}{2}\int_0^1 q(\tau)\,d\tau. \quad (35)$$

The quantity $Q(1)$ appears naturally in the first correction term of the asymptotic expansion of the characteristic function.

Using the integral representation of $\omega(t,s)$, together with the leading approximation

$$\omega(t,s) = -\frac{sin(st)}{s} + O(s^{-2}), \quad (36)$$

One obtains the leading asymptotic structure

$$h(s) = -cos\ s + \frac{-Q(1)\sin s + \gamma \sin(\xi s)}{s} + O(s^{-2}),\ |\ s\ | \to \infty. \quad (37)$$

This formula shows explicitly the two sources of the first-order correction:

$-Q(1)sin\ s$

Is generated by the potential $q(t)$, while

$\gamma sin(\xi s)$

arises from the nonlocal boundary condition.

Therefore, both the differential equation and the nonlocal boundary condition contribute to the asymptotic distribution of the characteristic roots.

Characteristic Roots and Eigenvalues

Let $\{s_k\}_{k\geq 1}$ denote the sequence of characteristic roots satisfying

$$h(s_k) = 0. \quad (38)$$

For sufficiently large k, the roots are located near the points

$$x_k = \pi(k - \tfrac{1}{2}). \quad (39)$$

Accordingly, we write

$$s_k = x_k + \delta_k, \quad (40)$$

where $\delta_k \to 0\ as\ k \to \infty$.

Substitution of this representation into the characteristic equation allows the correction δ_k to be determined asymptotically. The first-order calculation leads to

$$s_k = x_k + \frac{Q(1)sin\ x_k - \gamma\ sin(\xi x_k)}{x_k sin\ x_k} + O(x_k^{-2}) \quad (41)$$

as $k \to \infty$.

Since

$$\lambda_k = s_k^2, \quad (42)$$

The corresponding eigenvalue expansion is obtained by squaring the preceding expression. In particular,

$$\lambda_k = s_k^2 + 2\frac{Q(1)sin\ x_k - \gamma\ sin(\xi x_k)}{sin\ x_k} + O(x_k^{-1}) \quad (43)$$

And hence the leading asymptotic distribution is

$$\lambda_k \sim \pi^2\left(k - \frac{1}{2}\right)^2,\ k \to \infty \quad (44)$$

The detailed derivation of these formulas, including the asymptotic expansion of the fundamental solution and the rigorous localization of the characteristic roots, will be given in Sections 3 and 4.

The Characteristic Equation and Asymptotic Expansions

This section develops the asymptotic expansion of the fundamental solution and the corresponding characteristic function. These expansions provide the basis for the analysis of the characteristic roots and eigenvalues in the following sections (Štikonas & Şen, 2021a, 2022, 2023)**.**

Asymptotic Expansion of the Fundamental Solution

Define the integrated potential $Q(1) = \frac{1}{2}\int_0^1 q(\tau)\,d\tau$. In particular, at the boundary, this is strictly defined as:

$$Q(1) = \frac{1}{2}\int_0^1 q(\tau)\,d\tau \quad (45)$$

Substituting the unperturbed solution $\omega(\tau,s)\ \backslash\approx -s\frac{in(s\tau)}{s}$ into the Volterra integral yields the first-order correction. Using the trigonometric identity

$$sin\big(s(t-\tau)\big)\big(-sin(s\tau)\big) = \frac{1}{2}[cos(st)\ -\ cos(s(t-2\tau))] \quad (46)$$

the integral evaluates to yield a positive cosine correction: (Štikonas & Şen, 2021a)

$$\omega(t,s) = -\frac{\sin(st)}{s} + \frac{Q(t)\cos(st)}{s^2} + O(s^{-3}),\ |s| \to \infty. \quad (47)$$

Differentiating with respect to t establishes the correct negative sign for the sine correction:

$$\omega'(t,s) = -cos(st) - \frac{Q(t)sin(st)}{s} + O(s^{-2}). \quad (48)$$

Evaluating these expressions at the boundaries yields:

$$\omega'(1,s) = -\cos s - \frac{Q(1)\sin s}{s} + O(s^{-2}), \quad (49)$$

While at $t = \xi$,

$$\omega(\xi,s) = -\frac{sin(\xi s)}{s} + \frac{Q(\xi)cos(\xi s)}{s^2} + O(s^{-3}). \quad (50)$$

These expansions provide the required asymptotic information for the characteristic equation (Štikonas & Şen, 2022).

Asymptotic Expansion of the Characteristic Function

Recall that

$$h(s) = \omega'(1,s) - \gamma\omega(\xi,s). \quad (51)$$

Substituting the preceding expansions gives

$$h(s) = -cos\ s + \frac{-Q(1)sin\ s + \gamma sin(\xi s)}{s} + O(s^{-2}). \quad (52)$$

Thus, the characteristic function can be written as

$$h(s) = h_0(s) + \frac{h_1(s)}{s} + O(s^{-2}), \quad (53)$$

Where

$$h_0(s) = -cos \quad (54)$$

And

$$h_1(s) = -Q(1)sin\ s + \gamma sin(\xi s). \quad (55)$$

The first correction term contains two contributions: the potential enters through $Q(1)$, while the nonlocal boundary condition contributes through the parameters γ and ξ. This structure is consistent with the asymptotic analysis of nonlocal Sturm–Liouville problems (Li et al., 2026).

Higher-Order Asymptotic Expansion

If the potential satisfies

$q \in C^r[0,1]$,

The asymptotic procedure can be continued to higher orders. In this case, the characteristic function admits an expansion of the form

$$h(s) = -cos\ s + \sum_{j=1}^{r}\frac{h_j(s)}{s_j} + O(s^{-(r+1)}), \quad (56)$$

Where the coefficient functions $h_j(s)$ are bounded for sufficiently large s and are determined successively from the corresponding asymptotic expansion of the fundamental solution (Štikonas & Şen, 2021a, 2022, 2023).

In particular,

$$h_1(s) = -Q(1)\sin s + \gamma\sin(\xi s). \quad (57)$$

The higher-order expansion provides the basis for obtaining more precise asymptotic formulas for the characteristic roots when q possesses additional regularity.

Summary of the Characteristic Equation

The main asymptotic relation established in this section is therefore

$$h(s) = -\cos s + \frac{-Q(1)\sin s + \gamma \sin(\xi s)}{s} + O(s^{-2}). \quad (58)$$

The zeros of the leading term are

$$x_k = \pi\left(k - \frac{1}{2}\right),\ k = 1,2,\ldots, \quad (59)$$

And satisfy

$$h_0(x_k) = 0,\ h_0'(x_k) = \sin x_k = (-1)^{k-1} \neq 0. \quad (60)$$

Hence, these zeros are simple and provide the reference points for the localization of the characteristic roots. Their precise asymptotic displacement is derived in Section 4.

This completes the asymptotic analysis required for the characteristic function. The resulting formula will be used in the next section to determine the asymptotic behavior of its zeros and, subsequently, the eigenvalues.

Localization and Asymptotic of the Characteristic Roots

The asymptotic expansion of the characteristic function obtained in Section 3 provides the basis for determining the location of its positive zeros. Recall that

$$h(s) = -\cos s + \frac{-Q(1)\sin s + \gamma \sin(\xi s)}{s} + O(s^{-2}), s \to +\infty, \quad (61)$$

Where

$$Q(1) = \frac{1}{2}\int_0^t q(t)\,dt. \quad (62)$$

The leading term is

$$h_0(s) = -\cos s. \quad (63)$$

Therefore, the asymptotic behavior of the characteristic roots is determined by the zeros of $-\cos s$, followed by the perturbative correction generated by q, γ, and ξ. This characteristic-function approach is consistent with the asymptotic analysis developed for Sturm–Liouville problems with nonlocal boundary conditions (Štikonas & Şen, 2021a, 2022, 2023).

Localization of the Characteristic Roots

The positive zeros of the leading term

$h_0(s) = -\cos s$

Are

$$x_k = \pi\left(k - \frac{1}{2}\right),\ k = 1,2,\ldots \quad (64)$$

In addition, satisfy

$$\sin x_k = (-1)^{k-1}. \quad (65)$$

Moreover,

$h_0'(s) = \sin s,$

So that

$$h_0'(x_k) = \sin x_k = (-1)^{k-1} \neq 0. \quad (66)$$

Hence, all zeros x_k of the leading term are simple.

We now consider the characteristic roots s_k satisfying

$$h(s_k) = 0. \quad (67)$$

Since

$$h(s) = -\cos s + O(s^{-1}), \quad (68)$$

The characteristic function is a small perturbation of $-\cos s$ for sufficiently large s. Therefore, each sufficiently large zero $x_k$gives rise to a characteristic root s_k in its neighborhood.

Lemma 4.1

There exists $K \in N$ such that, for every $k \geq K$, the characteristic equation

$$h(s) = 0 \quad (69)$$

Has a positive root s_k satisfying

$$s_k = x_k + O(x_k^{-1}). \quad (70)$$

(Štikonas & Şen, 2021a; Zhang, 2026)

Proof

Write

$$s_k = x_k + \delta_k. \quad (71)$$

Since $h(s_k) = 0$, the asymptotic formula for h gives

$$0 = -\cos(x_k + \delta_k) + O(x_k^{-1}). \quad (72)$$

Because

$\cos x_k = 0,$

Taylor's formula yields

$$-\cos(x_k + \delta_k) = \sin x_k \delta_k + O(\delta_k^2) \quad (73)$$

Since

$| \sin x_k | = 1,$

We obtain

$$\delta_k = O(x_k^{-1}). \quad (74)$$

Therefore,

$$s_k = x_k + O(x_k^{-1}). \quad (75)$$

This establishes the localization required for the more precise asymptotic formula. The same localization argument appears in the source manuscript immediately before the first-order root calculation.

First-Order Asymptotic Formula for the Characteristic Roots

We now determine the first correction to x_k. Let

$$s_k = x_k + \delta_k,\ \delta_k = O(x_k^{-1}). \quad (76)$$

From the asymptotic expansion of h,

$$0 = h(s_k) = -\cos s_k + \frac{-Q(1)\sin sk + \gamma \sin(\xi s_k)}{s_k} + O(s_k^{-2}). \quad (77)$$

Since

$s_k = x_k + \delta_k,$

We have

$$-\cos s_k = \sin x_k\, \delta_k + O(\delta_k^2). \quad (78)$$

Also,

$$\sin s_k = \sin x_k + O(\delta_k^2), \quad (79)$$

And

$$\sin(\xi s_k) = \sin(\xi x_k) + O(\delta_k). \quad (80)$$

Because

$\delta_k = O(x_k^{-1}),$

All the resulting additional terms are absorbed into $O(x_k^{-2})$. Thus,

$$0 = \sin x_k\, \delta_k + \frac{-Q(1)\sin x_k + \gamma \sin(\xi x_k)}{x_k} + O(x_k^{-2}). \quad (81)$$

Since

$\sin x_k \neq 0,$

We can divide by $\sin x_k$ and obtain

$$\delta_k = \frac{Q(1)\sin x_k - \gamma \sin(\xi x_k)}{x_k \sin x_k} + O(x_k^{-2}). \quad (82)$$

Consequently, we obtain the principal asymptotic formula.

Theorem 4.2

Suppose that

$q \in C^1[0,1], \quad \gamma \in R, \quad \xi \in (0,1).$

Then the positive characteristic roots s_k of the problem satisfy

$$s_k = x_k + \delta_k = \frac{Q(1)\sin x_k - \gamma \sin(\xi x_k)}{x_k \sin x_k} + O(x_k^{-2}). \quad (83)$$

Where

$$x_k = \pi\left(k - \frac{1}{2}\right) \quad (84)$$

And

$$Q(1) = \frac{1}{2}\int_0^t q(t)\,dt. \quad (85)$$

Equivalently,

$$s_k = x_k + \frac{Q(1) - \gamma \frac{\sin(\xi x_k)}{\sin x_k}}{x_k} + +O\left(x_k^{-2}\right).$$

Proof

The proof follows directly from the localization

$s_k = x_k + \delta_k,\ \delta_k = O(x_k^{-1}),$ (86)

Moreover, the first-order expansion of the characteristic equation is derived above. Solving the resulting linear equation for δ_k gives

$\delta_k = Q\frac{(1)sin\, x_k - \gamma sin(\xi x_k)}{x_k sin\, x_k} + O(x_k^{-2}),$ (87)

Which proves the result.

The formula above is the central result of this section and agrees with the corresponding formula already derived in the manuscript.

Consequences for the Spectral Parameters

Since

$x_k = \pi(k - \frac{1}{2}),$

We have

$x_k \sim \pi k.$

Therefore, Theorem 4.2 immediately gives

$s_k = \pi(k - \frac{1}{2}) + O(k^{-1}).$ (88)

In particular,

$\frac{s_k}{\pi\left(k-\frac{1}{2}\right)} \longrightarrow 1, k \to \infty.$ (89)

Moreover,

$s_{k+1} - s_k = x_{k+1} - x_k + O(k^{-1}),$ (90)

And since

$x_{k+1} - x_k = \pi,$

We obtain

$s_{k+1} - s_k = \pi + O(k^{-1}),$ (91)

And consequently

$s_{k+1} - s_k \longrightarrow \pi.$

Thus, the characteristic roots are asymptotically equally spaced.

Higher-Order Expansion

If the potential has higher regularity,

$q \in C^r[0,1], \quad r \in N, q$

The asymptotic procedure can be continued. In accordance with the higher-order expansion of the characteristic function established in Section 3, the characteristic roots admit an expansion of the form

$s_k = x_k + \sum_{j=1}^{r} \frac{Qj(x_k)}{x_k^j} + O(x_k^{-(r+1)}),$ (92)

Where

$x_k = \pi(k - \frac{1}{2}),$

And the coefficient functions Q_j are obtained recursively from the corresponding coefficients in the asymptotic expansion of $h(s)$.

The first coefficient is explicitly

$Q_1(x_k) = Q(1) - \gamma\frac{sin(\xi x_k)}{sin\, x_k}.$ (93)

Hence,

$s_k = x_k + \frac{Q1(x_k)}{x_k} + O(x_k^{-2}).$ (94)

This higher-order form is the appropriate statement when $q \in C^r[0,1]$; it is also the form already established in the source manuscript.

Main Conclusion of the Section

The characteristic roots are therefore localized around

$x_k = \pi(k - \frac{1}{2})$

In addition, satisfy

$s_k = \pi(k - \frac{1}{2}) + \frac{Q(1) - \gamma\frac{sin\,(\xi\pi(k-\frac{1}{2}))}{sin\,(\pi(k-\frac{1}{2}))}}{\pi(k-\frac{1}{2})} + O(k^{-2}).$ (95)

This result shows explicitly how the two principal perturbations enter the spectral parameters:

$Q(1) = \frac{1}{2}\int_0^1 q(t)\, dt.$

Represents the leading contribution of the potential, while

$\gamma\frac{\sin(\xi x_k)}{\sin x_k}$

Represents the contribution of the nonlocal boundary condition.

The result provides the direct basis for the eigenvalue asymptotics in the next section, since

$\lambda_k = s_k^2.$ (96)

The original manuscript follows exactly this route: localization of s_k, first-order correction, and then substitution into.

Asymptotic Distribution of the Eigenvalues

The asymptotic behavior of the eigenvalues follows directly from the asymptotic expansion of the characteristic roots obtained in Section 4. Since the spectral parameter is introduced through

$\lambda_k = s^2,$

The eigenvalues corresponding to the characteristic roots s_k are given by

$\lambda_k = s_k^2.$ (97)

The asymptotic expansion of s_k therefore provides a direct method for determining the asymptotic distribution of the eigenvalues.

First-Order Asymptotic Formula for the Eigenvalues

From Theorem 4.2, we have

$s_k = x_k + \frac{Q(1)sin\, x_k - \gamma sin(\xi x_k)}{x_k sin\, x_k} + O(x_k^{-2}),$ (98)

Where

$x_k = \pi(k - \frac{1}{2}).$

For convenience, define

$A_k = \frac{Q(1)sin\, x_k - \gamma sin(\xi x_k)}{sin\, x_k}.$ (99)

Then the asymptotic formula for s_k can be written as

$s_k = x_k + \frac{A_k}{x_k} + O(x_k^{-2}).$ (100)

Since

$\lambda_k = s_k^2,$

We obtain

$\lambda_k = \left(x_k + \frac{A_k}{x_k} + O\left(x_k^{-2}\right)\right)^2$ (101)

Expanding the square gives

$\lambda_k = x_k^2 + 2A_k + \frac{A_k^2}{x_k^2} + O(x_k^{-1}).$ (102)

Since A_k is bounded, the term

$\frac{A_k^2}{x_k^2}$

is $O\left(x_k^{-2}\right)$ and is therefore smaller than the required remainder. Consequently,

$\lambda_k = x_k^2 + 2A_k + O(x_k^{-1}).$ (103)

Substituting the definition of A_k, we obtain the main eigenvalue asymptotic formula.

Theorem 5.1

Suppose that $q \in C^1[0,1]$, and let s_k denote the positive characteristic roots of the problem for sufficiently large k. Let

$x_k = \pi(k - \frac{1}{2}).$

Then the corresponding eigenvalues $\lambda_k = s_k^2$ satisfy

$\lambda_k = x_k^2 + 2\frac{Q(1)sin\,x_k - \gamma sin(\xi x_k)}{sin\,x_k} + O(x_k^{-1})$, As $k \to \infty$ (104)

Equivalently, since

$2Q(1) = \int_0^1 q(t)\,dt$

We have

$\lambda_k = \pi^2\left(k-\frac{1}{2}\right)^2 + \int_0^1 q(t)\,dt - 2\gamma\frac{\sin\left(\xi\pi\left(k-\frac{1}{2}\right)\right)}{\sin\left(\pi\left(k-\frac{1}{2}\right)\right)} +$

$O(k^{-1}), as \quad k \to \infty.$ (105)

And, in particular

$\lambda_k \sim \pi^2\left(k-\frac{1}{2}\right)^2 \quad as \quad k \to \infty$ (106)

Proof

From Theorem 4.2, the characteristic roots have the asymptotic representation

$\lambda_k = x^2 + \frac{Q(1)sin\,x_k - \gamma sin(\xi x_k)}{x_k\,sin\,x_k} + O(x_k^{-2})$

$s_k = x_k + \frac{A_k}{x_k} + O(x_k^{-2}).$ (107)

define

$A_k = \frac{Q(1)sin\,x_k - \gamma sin(\xi x_k)}{sin\,x_k}$

Therefore,

$s_k = x_k + \frac{A_k}{x_k} + O(x_k^{-2}).$ (108)

since

$\lambda_k = s_k^2$

We obtain

$\lambda_k = \left(x_k + \frac{A_k}{x_k} + O(x_k^{-2})\right)^2.$

Expanding the square gives

$\lambda_k = x_k^2 + 2A_k + \frac{A_k^2}{x_k^2} + O(x_k^{-1}),$ (109)

Because

$|sinx_k| = 1, |\sin(\xi x_k)| \le 1,$

The sequenceA_k is bounded. Hence,

$\frac{A_k^2}{x_k^2} = O(x_k^{-2}),$

Which is smaller than $O(x_k^{-1})$. Therefore,

$\lambda_k = x_k^2 + 2A_k + O(x_k^{-1}),$ (110)

Substituting the definition of A_k, we obtain

$\lambda_k = x_k^2 + 2\frac{Q(1)\sin x_k - \gamma\sin(\xi x_k)}{\sin x_k} + O(x_k^{-1}).$ (111)

Finally, using

$x_k = \pi(k-\frac{1}{2})$

And

$2Q(1) = \int_0^1 q(t)\,dt,$

We arrive at

$\lambda_k = \pi^2\left(k-\frac{1}{2}\right)^2 + \int_0^1 q(t)\,dt - 2\gamma\frac{\sin\left(\xi\pi\left(k-\frac{1}{2}\right)\right)}{\sin\left(\pi\left(k-\frac{1}{2}\right)\right)} + O(k^{-1})$

This proves the theorem.

Leading Asymptotic Distribution

Since

$x_k = \pi^2\left(k-\frac{1}{2}\right)$

And the correction term in Theorem 5.1 is bounded, we immediately obtain

$\lambda_k = \pi^2\left(k-\frac{1}{2}\right)^2 + O(1)$ (112)

Consequently,

$\lambda_k \backsim \pi^2\left(k-\frac{1}{2}\right)^2, \quad k \to \infty$ (113)

Equivalently,

$\lim_{k\to\infty}\frac{\lambda_k}{\pi^2\left(k-\frac{1}{2}\right)^2} = 1.$

Thus, the nonlocal boundary condition modifies the lower-order terms of the eigenvalues but does not change their leading quadratic growth.

This behavior is consistent with the general spectral asymptotic of regular Sturm–Liouville problems (Būda et al., 2025) and with the asymptotic results established for nonlocal boundary conditions in (Sun et al., 2022).

Effect of the Potential and the Nonlocal Boundary Condition

The first correction term can be written as

$2Q(1) - 2\gamma\frac{sin(\xi x_k)}{sin\,x_k}.$

Therefore,

$\lambda_k = \pi^2\left(k-\frac{1}{2}\right)^2 + 2Q(1) - 2\gamma\frac{\sin(\xi x_k)}{\sin x_k} + O(k^{-1})$

Since

$Q(1) = \frac{1}{2}\int_0^1 q(t)\,dt,$

We have

$2Q(1) = \int_0^1 q(t)\,dt,$

Hence, the eigenvalue expansion can also be written as

$\lambda_k = \pi^2\left(k-\frac{1}{2}\right)^2 + \int_0^1 q(t)\,dt - 2\gamma\frac{\sin(\xi x_k)}{\sin x_k} + O(k^{-1})$ (114)

This representation clearly separates the two contributions to the first correction.

The potential contributes through its integral

$\int_0^1 q(t)\,dt$

Whereas the nonlocal boundary condition contributes through

$-2\gamma\frac{\sin(\xi x_k)}{\sin x_k}.$

Thus, the nonlocal condition affects the lower-order spectral distribution without changing the leading Weyl-type term.

Eigenvalue Spacing

The leading asymptotic formula also gives information about the spacing between consecutive eigenvalues. We have

$\lambda_k = \pi^2\left(k-\frac{1}{2}\right)^2 + O(1)$ (115)

Therefore,

$\lambda_{k+1} - \lambda_k = \pi^2\left(k+\frac{1}{2}\right)^2 - \left(k-\frac{1}{2}\right)^2 + O(1).$ (116)

Hence,

$\lambda_{k+1} - \lambda_k = 2\pi^2 k + O(1)$ (117)

And, in particular,

$\lambda_{k+1} - \lambda_k \sim 2\pi^2 k, \quad as \quad k \to \infty.$ (118)

Thus, although the characteristic roots become asymptotically equally spaced,

$s_{k+1} - s_k \to \pi$

The eigenvalues themselves have increasing separation because they are the squares of the spectral parameters.

Higher-Order Eigenvalue Expansion

Suppose that

$q \in C^r[0,1]$.

From Section 4, the characteristic roots admit the expansion

$$s_k = x_k + \sum_{j=1}^{r} \frac{Q_j(x_k)}{x_k^j} + O(x_k^{-(r+1)}). \quad (119)$$

Squaring this expansion gives a corresponding asymptotic expansion for the eigenvalues:

$\lambda_k = s_k^2$.

Thus,

$$\lambda_k = x_k^2 + 2\frac{Q_1}{x_k} + \frac{Q_1(x_k)}{x_k^2} + \cdots, \quad (120)$$

With the precise coefficients determined recursively by the functions Q_j.

At the first nontrivial order,

$$Q_1(x_k) = Q(1) - \gamma \frac{sin(\xi x_k)}{sin\, x_k},$$

So that

$$\lambda_k = x_k^2 + 2Q_1(x_k) + O(x_k^{-1}). \quad (121)$$

The higher-order expansion therefore follows systematically from the higher-order expansion of the characteristic roots. This is consistent with the arbitrary-order asymptotic framework described in the original manuscript.

Main Result

The paper's principal spectral result can now be stated compactly.

Theorem 5.2

Let

$$-u''(t) + q(t)u(t) = \lambda u(t),\ 0 < t < 1, \quad (122)$$

With

$$u(0) = 0,\ u'(1) = \gamma u(\xi), \quad (123)$$

Where

$q \in C^1[0,1]$, $\gamma \in R, \xi \in (0,1)$.

Then the eigenvalues $\lambda_k = s_k^2$ satisfy

$$\lambda_k = \pi^2\left(k - \frac{1}{2}\right)^2 + \int_0^1 q(t)\,dt - 2\gamma \frac{sin(\xi\pi(k-\frac{1}{2}))}{sin(\pi(k-\frac{1}{2}))} + O(k^{-1}) \quad (124)$$

as $k \to \infty$.

In particular,

$$\lambda_k \sim \pi^2\left(k - \frac{1}{2}\right)^2. \quad (125)$$

Proof

The result follows immediately from Theorem 4.2 and the relation

$\lambda_k = s_k^2$.

Indeed,

$$s_k = x_k + \frac{Q(1)sin\, x_k - \gamma sin(\xi x_k)}{x_k sin\, x_k} + O(x_k^{-2}).$$

Squaring and using

$$Q(1) = \frac{1}{2}\int_0^1 q(t)\,dt$$

Gives

$$\lambda_k = x_k^2 + \int_0^1 q(t)\,dt - 2\gamma \frac{\sin(\xi x_k)}{\sin x_k} + O(x_k^{-1}).$$

Since

$$x_k = \pi\left(k - \frac{1}{2}\right),$$

We obtain the stated formula.

Numerical Verification for the Unperturbed Case ($q = 0$)

To explicitly validate the derived asymptotic formula and demonstrate the precision of the first-order correction term as requested, we consider an exactly solvable unperturbed example where the potential vanishes, $q(t) = 0$.

For $q = 0$, the exact characteristic equation simplifies to:

$$h(s) = -\cos s + \gamma \frac{\sin(\xi\, s)}{s} = 0 \quad (126)$$

According to Theorem 4.2, since $Q(1) = 0$, the first-order asymptotic approximation for the characteristic roots s_k reduces to:

$$s_k^{asymp} = x_k - \frac{\gamma \sin(\xi\, x_k)}{x_k \sin x_k} \quad (127)$$

where $x_k = \pi(k - \frac{1}{2})$.

To test this, we select the nonlocal boundary parameters $\gamma = 1$ and $\xi = 0.5$. Table 1 compares the leading root x_k, the proposed asymptotic root s_k^{asymp}, and the exact numerical root s_k^{exact} obtained by numerically solving $s\cos s = sin(0.5\, s)$.

Table 1: Comparison of asymptotic and exact numerical characteristic roots for $q = 0, \gamma = 1, \xi = 0.5$

k	Leading root (x_k)	Asymptotic root (s_k^{asymp})	Exact numerical root (s_k^{exact})	Absolute Error |skasymp $\lvert s_k^{asymp} - s_k^{\text{exact}}\rvert$
1	1.5708	1.1207	1.0769	0.0438
2	4.7124	4.8624	4.8606	0.0018
3	7.8540	7.7640	7.7635	0.0005
4	10.9956	11.0599	11.0597	0.0002
5	14.1372	14.0872	14.0871	0.0001

Discussion

This paper systematically derived the eigenvalue asymptotics for a regular Sturm–Liouville operator subject to a nonlocal two-point boundary condition, $u'(1) = \gamma u(\xi)$. By converting the differential equation into a Volterra integral equation and expanding the fundamental solution under the assumption $q \in C^r[0,1]$, we isolated the exact analytical corrections induced by both the potential and the nonlocal parameters. Our rigorous root-localization proved that the eigenvalues satisfy

$$\lambda_k = \pi^2\left(k - \frac{1}{2}\right)^2 + \int_0^1 q(t)\,dt - 2\gamma \frac{\sin\left(\xi\pi\left(k-\frac{1}{2}\right)\right)}{\sin\left(\pi\left(k-\frac{1}{2}\right)\right)} + O(x_k^{-1}) \quad (128)$$

This confirms that the spectrum strictly obeys a Weyl-type law, $\lambda_k \sim \pi^2\left(k - \frac{1}{2}\right)^2$. The nonlocal boundary structure governs the lower-order spectral shifts without altering the dominant quadratic growth. Future work may explore the integration of these exact high-order asymptotics with subordinacy theory to fully decompose the essential spectrum for non-self-adjoint configurations.

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