

Stability and Benchmark-Based Convergence Assessment of Extended Cubic B-Spline Collocation for Time-Fractional Advection-Diffusion Equations

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ABSTRACT

This study examines a fully implicit finite difference scheme based on extended cubic B-spline collocation for solving one-dimensional time-fractional advection–diffusion equations. The time-fractional derivative is approximated using the Caputo formulation with the L1 discretization scheme, while the spatial derivatives are approximated using extended cubic B-spline basis functions. The resulting fully discrete system incorporates the memory effects of the fractional derivative together with the prescribed initial and boundary conditions. The stability and convergence properties of the scheme are examined through Fourier analysis and mathematical approximation estimates under appropriate assumptions. The numerical performance is assessed using two benchmark problems with known analytical solutions. Comparisons of the published numerical and exact solutions demonstrate good agreement, while the reported error norms and spatial refinement results indicate decreasing numerical errors as the spatial grid is refined. The findings support the accuracy and applicability of the fully implicit extended cubic B-spline collocation scheme for the considered benchmark problems.

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INTRODUCTION

Advection diffusion equations describe transport governed by directed motion and spatial dispersion. In heterogeneous systems, however, anomalous spreading, delayed migration and memory effects may require nonlocal temporal operators (Khan, 2026, p. 9; Lin et al., 2023, p. 129433). Fractional models provide a framework for hereditary responses and nonclassical transport (Alshehry et al., 2026, p. 2; Pleszczyński, 2026; Hristov, 2023). Their applications encompass non-Gaussian transport and departures from classical diffusion (Suzuki et al., 2021, p. 4; Podlubný, 2025, p. 63). In the time-fractional advection diffusion equation, a Caputo derivative replaces the classical first-order time derivative while preserving conventional initial conditions; its history dependence, however, increases the cost and complexity of numerical integration.

Numerical treatments of fractional transport include finite element, finite difference and finite volume formulations. Variational discretizations address nonlocal spatial operators (Jin et al., 2014, p. 2273), while characteristic and implicit difference approaches have been considered for advection–diffusion systems (Bibi & Ejaz, 2026, p. 3; Bibi & Ejaz, 2026, p. 2). Finite volume methods (V, 2025), finite element approaches to nonlinear fractional transport (Mehri et al., 2025), rational approximation techniques (Margenov, 2026, p. 343) and mixed finite element schemes (Yang et al., 2026) broaden the available computational strategies. Analytical solutions and operator-based formulations supply complementary benchmarks and modelling perspectives (Khan, 2026, p. 15; Allagui et al., 2024, p. 12888; Alshehry et al., 2025).

B-spline collocation offers a local, continuous representation of the numerical solution and its spatial derivatives (Reddy, 2025). Extended cubic B-splines introduce a parameter that

changes the nodal basis coefficients and derivative approximations, recovering the conventional cubic basis when the extension parameter vanishes (Krishnamohan, 2025). Related spline methods have been applied to fractional diffusion, diffusion–wave systems (Morsi et al., 2023, p. 836; Shafiq et al., 2025), fractional subdiffusion (Yaseen et al., 2021, p. 2), convection–diffusion equations (Roul & Goura, 2020), and fractional evolution problems using trigonometric bases or differential quadrature (Yaseen et al., 2020; Manzoor et al., 2025). These studies support the flexibility of spline approximations but do not establish that any one basis is uniformly superior.

Recent investigations published in the FUDMA Journal of Sciences demonstrate the continuing development of numerical approximation methods for fractional differential equations and related mathematical models. Uwaheren et al. (2025) examined an exponential-fitted collocation approach for fractional integro-differential equations, highlighting the application of structured collocation techniques to nonlocal mathematical problems. Olayiwola et al. (2025) employed Hermite polynomial collocation and Laplace–Adomian decomposition in analysing a Caputo fractional-order model, illustrating the usefulness of numerical approximation for systems exhibiting memory-dependent dynamics. Similarly, Okunola et al. (2025) developed a Laguerre-perturbed Galerkin method for higher-order nonlinear integro-differential equations and assessed its numerical accuracy using benchmark problems. These contributions establish a relevant methodological context for the present investigation, particularly its emphasis on fractional discretization, spline-based approximation, stability conditions, and benchmark-based convergence assessment.

Approximation accuracy depends on the spatial operator, boundary treatment and regularity of the exact solution.

Analyses of fractional Laplacian discretizations and high-order finite differences illustrate distinct routes to error control (Ainsworth & Glusa, 2017, p. 5; Chen & Deng, 2014, p. 1418). Further applications of spline and related approximation techniques include fractional-order circuit models (Zambrano-Serrano et al., 2021, p. 31), singularly perturbed boundary-value problems (Arumugam et al., 2025), and nonlinear equations treated with cubic B-splines (Yaseen et al., 2025). The latter citation also appears in the source in connection with spline approximation and is retained here without attributing a separate convergence theorem to it (Yaseen et al., 2025).

The fully implicit extended cubic B-spline formulation considered here combines an L_1 discretization of the Caputo derivative with spline collocation for the advective and diffusive terms. The source manuscript associates this established approach with Mohyud-Din et al. but cites Podlubný (2025, p. 197); this attribution requires bibliographic confirmation. The present study focuses on evaluating the mathematical formulation and numerical performance of the fully implicit extended cubic B-spline collocation scheme for time-fractional advection–diffusion equations. Particular attention is given to the consistency of the numerical discretization, the conditions governing stability and convergence, and the accuracy demonstrated by published benchmark results.

The assessment also considers the agreement between numerical and analytical solutions and the influence of spatial refinement on numerical errors, while identifying limitations in the available evidence concerning temporal convergence and computational efficiency.

Temporal accuracy requires separate scrutiny because solutions of fractional evolution equations may lack smoothness near the initial time. Analyses of nonsmooth data and singular source terms show why formal $L1$ consistency does not by itself establish an observed convergence rate (Zhou & Tian, 2022, p. 2; Li et al., 2024, p. 430). Higher-order Caputo-type methods (Shams & Carpentieri, 2025), fractional stability techniques (Du et al., 2020, p. 3), convolution quadrature and fully discrete schemes for nonsmooth data (Jin et al., 2016, p. 3; Cuesta et al., 2006, p. 673) provide relevant points of comparison. For sufficiently regular solutions, the $L1$ method has formal temporal consistency $O(\tau^\gamma)$; a combined $O(h^2 + \tau^\gamma)$ estimate additionally requires suitable spatial approximation and stability assumptions, and must not be treated as experimentally established without refinement evidence.

Two analytical benchmarks underpin the source-based assessment: a solution with spatial exponential and Mittag-Leffler temporal dependence, and a time-dependent sinusoidal solution. Reported maximum absolute and discrete L_2 errors are used to assess agreement, while available spatial refinement results permit only a limited convergence appraisal. Independent temporal refinement, controlled variation of the spline extension parameter λ , and computational cost comparisons remain necessary before stronger accuracy or efficiency claims can be made. Accordingly, Section 2 presents the governing problem and collocation discretization, Section 3 examines stability and convergence conditions, and Section 4 discusses benchmark verification and limitations.

Mathematical Formulation

Governing Time-Fractional Equation

The governing model is the one-dimensional time-fractional advection–diffusion equation, in which the temporal evolution is represented by a Caputo derivative of fractional

order α . Writing v_s and v_{ss} for the first and second spatial derivatives, respectively, the equation is expressed as

$$D_{\theta}^{\alpha, c} v + c v_s - D v_{ss} = F(s, \theta) \quad (1)$$

Here, $0 < \alpha < 1$ denotes the fractional order, c is the constant advection velocity, $D > 0$ is the diffusion coefficient, and $F(s, \theta)$ is the prescribed source function. The spatial coordinate s belongs to the interval $[L, R]$, while θ denotes time over $(0, \Theta]$. The dependent variable $v(s, \theta)$ represents the transported quantity. This formulation incorporates advective transport, diffusion, and temporal memory within a common mathematical model (Mohyud-Din et al., 2018).

Initial and Boundary Conditions

The problem is completed by prescribing an initial state and Dirichlet boundary conditions at the endpoints of the spatial domain. The initial condition is

$$v(s, 0) = v_0(s), \quad L \leq s \leq R \quad (2)$$

The boundary conditions are

$$v(L, \theta) = \ell(\theta), \quad v(R, \theta) = \rho(\theta) \quad (3)$$

The functions $v_0(s)$, $\ell(\theta)$, and $\rho(\theta)$ specify the initial profile and the left- and right-boundary data, respectively. Compatibility at the initial time requires $v_0(L) = \ell(0)$ and $v_0(R) = \rho(0)$. In the collocation formulation, these conditions determine the boundary nodal equations and support construction of the initial spline coefficients.

Caputo Fractional Derivative

For $0 < \alpha < 1$, the Caputo fractional derivative with respect to time, denoted by D with superscripts α and C , is defined by

$$D_{\theta}^{\alpha, c} v = \int_0^{\theta} K_{\alpha}(\theta - \zeta) v_{\zeta}(s, \zeta) d\zeta \quad (4)$$

In Equation (4), the memory kernel is $K_{\alpha}(t) = t^{-\alpha} \Gamma(1 - \alpha)$ for $t > 0$; $\Gamma(\cdot)$ denotes the Gamma function, ζ is the integration variable, and $v_{\zeta}(s, \zeta)$ denotes differentiation with respect to ζ . The definition requires suitable temporal regularity, such as absolute continuity of $v(s, \cdot)$ on the relevant time interval, so that the integral is well defined. Its nonlocal kernel accounts for the contribution of preceding states to the current evolution. This memory dependence motivates the $L1$ convolution approximation employed in the temporal discretization in Section 3 (Mohyud-Din et al., 2018).

Numerical Discretization and Collocation Scheme

Temporal discretization using the $L1$ scheme

Let $s_j = L + j\Delta s$ ($j = 0, \dots, J$), where $\Delta s = (R - L)/J$, and $\theta_n = n\Delta\theta$ ($n = 0, \dots, K$), with $\Delta\theta = \Theta/K$. Denote by V_j^n the approximation of $v(s_j, \theta_n)$. The $L1$ approximation of the Caputo derivative at $\theta(n+1)$ is

$${}^C D_{\theta}^{\alpha, c} v(s_j, \theta_{n+1}) \approx \mu \sum_{r=0}^n w_r (V_j^{n+1-r} - V_j^{n-r}) \quad (5)$$

Here $\mu = 1/[\Gamma(2-\alpha)(\Delta\theta)^{\alpha}]$. The convolution weights are defined by

$$w_r = (r+1)^{1-\alpha} - r^{1-\alpha} \\ w_0 = 1, \quad w_r > 0, \quad w_r > w_{r+1} \quad (r \geq 0) \quad (6)$$

For $0 < \alpha < 1$, these weights are positive, strictly decreasing, and capture the contribution from all preceding time levels. Provided v has sufficiently regular second-order temporal derivatives, the $L1$ local truncation error is $O((\Delta\theta)^{\alpha}(2-\alpha))$. Reduced temporal regularity near $\theta = 0$ can affect this estimate and should be considered when assessing convergence.

Extended cubic B-spline basis functions

The spatial approximation uses compactly supported extended cubic B-spline functions $\psi_j(s; \kappa)$ on a uniform grid, with $h = \Delta s$. Their support is $[s_{j-2}, s(j+2)]$. Writing

$z = s - s_{j-1}$ and $y = s_{j+1} - s$, the piecewise representation is

$$\begin{aligned}\psi_j(s; \kappa) &= \frac{P_i(s)}{24h^4} \quad (s \in I_i), \quad i = 1, 2, 3, 4 \\ P_1(s) &= 4h(1 - \kappa)(s - s_{j-2})^3 + 3\kappa(s - s_{j-2})^4 \\ P_2(s) &= (4 - \kappa)h^4 + 12h^3z \\ &\quad + 6h^2(2 + \kappa)z^2 - 12hz^3 - 3\kappa z^4 \\ z &= s - s_{j-1} \\ P_3(s) &= (4 - \kappa)h^4 + 12h^3y \\ &\quad + 6h^2(2 + \kappa)y^2 - 12hy^3 - 3\kappa y^4 \\ y &= s_{j+1} - s \\ P_4(s) &= 4h(1 - \kappa)(s_{j+2} - s)^3 + 3\kappa(s_{j+2} - s)^4 \\ \psi_j(s; \kappa) &= 0 \quad (s \notin [s_{j-2}, s_{j+2}])\end{aligned}\quad (7)$$

The interval assignments are $I_1 = [s_{j-2}, s(j-1))$, $I_2 = [s(j-1), s(j))$, $I_3 = [s(j), s(j+1))$, and $I_4 = [s(j+1), s_{j+2}]$. The admissible extension-parameter interval is $-8 \leq \kappa \leq 1$ for the stated nodal-weight properties, and $\kappa = 0$ recovers the conventional cubic B-spline basis. At each interior node, the three-point value and derivative stencils are determined by the constants

$$\begin{aligned}a &= \frac{4 - \kappa}{24}, \quad b = \frac{8 + \kappa}{12}, \quad g = \frac{1}{2h} \\ d &= \frac{2 + \kappa}{2h^2}, \quad e = -\frac{2 + \kappa}{h^2}\end{aligned}\quad (8)$$

For clarity, δ_{-}^{in} denotes the time-dependent spline expansion coefficient, whereas d and e in Equation (8) denote scalar second-derivative stencil coefficients. The approximate solution takes the form

$$V(s, \theta_n) = \sum_{j=-1}^{J+1} \delta_j^n \psi_j(s; \kappa) \quad (9)$$

Evaluating the basis and its first two spatial derivatives at the grid nodes gives

$$V_j^n = a\delta_{j-1}^n + b\delta_j^n + a\delta_{j+1}^n \quad (10)$$

$$(V_s)_j^n = -g\delta_{j-1}^n + g\delta_{j+1}^n \quad (11)$$

$$(V_{ss})_j^n = d\delta_{j-1}^n + e\delta_j^n + d\delta_{j+1}^n \quad (12)$$

The sign in Equation (11) follows the increasing left and decreasing right branches of the piecewise basis in Equation (7); this convention is used consistently in the advection coefficients below.

Construction of the fully implicit scheme

Let $\mu = [\Gamma(2 - \alpha)(\Delta\theta)^\alpha]^{(-1)}$. Collocation of the governing fractional advection–diffusion equation at s_j and $\theta(n+1)$, using the L_1 approximation, yields

$$\begin{aligned}\mu \sum_{r=0}^n w_r (V_j^{n+1-r} - V_j^{n-r}) \\ + c(V_s)_j^{n+1} - D(V_{ss})_j^{n+1} = F_j^{n+1}\end{aligned}\quad (13)$$

Substituting the nodal spline identities from Equations (10)–(12), the coefficients of the unknown expansion at time level $n+1$ become

$$\begin{aligned}A_- = \mu a - cg - Dd, \quad A_0 = \mu b - De \\ A_+ = \mu a + cg - Dd\end{aligned}\quad (14)$$

The resulting fully implicit collocation equation is

$$\begin{aligned}A_- \delta_{j-1}^{n+1} + A_0 \delta_j^{n+1} + A_+ \delta_{j+1}^{n+1} \\ = F_j^{n+1} + \mu [V_j^n - \sum_{r=1}^n w_r (V_j^{n+1-r} - V_j^{n-r})]\end{aligned}\quad (15)$$

An equivalent history representation is $H_j^n = w_n V_j^0 + \sum_{r=0}^{n-1} (w_r - w_{r+1})(n-1-r)(w_r - w_{r+1}) V_j^{n-r}$, with $H_j^0 = V_j^0$. Therefore, the right-hand side of Equation (15) may also be written as $F_j^{(n+1)} + \mu H_j^n$. This form is useful when assembling the linear system and preserves the full fractional-memory contribution.

Treatment of boundary and initial conditions

At each time level, the $J+1$ collocation relations contain $J+3$ spline coefficients. The two Dirichlet boundary conditions close the system through

$$a\delta_{-1}^{n+1} + b\delta_0^{n+1} + a\delta_1^{n+1} = l(\theta_{n+1}) \quad (16)$$

$$a\delta_{J-1}^{n+1} + b\delta_J^{n+1} + a\delta_{J+1}^{n+1} = \rho(\theta_{n+1}) \quad (17)$$

Arrange the coefficient vector and the assembled system as

$$\begin{aligned}\delta^{n+1} &= (\delta_{-1}^{n+1}, \delta_0^{n+1}, \dots, \delta_{J+1}^{n+1})^T \\ A\delta^{n+1} &= b^{n+1}\end{aligned}\quad (18)$$

Here A is the $(J+3) \times (J+3)$ banded collocation matrix. Its first and final rows impose Equations (16) and (17), while its interior rows contain the coefficients A_- , A_0 and A_+ from Equation (14). In matching row order, $b^{(n+1)}$ consists of the left boundary value $l(\theta(n+1))$, the $J+1$ interior entries $F_j(n+1) + \mu H_j^{(J=0, \dots, J)}$, and the right boundary value $\rho(\theta(n+1))$. Thus the system has the same number of independent equations as unknown spline coefficients, subject to nonsingularity of the assembled matrix.

The initial coefficient vector is determined from the $J+1$ nodal interpolation conditions and two endpoint-derivative conditions, assuming the initial profile v_0 is differentiable at both endpoints:

$$Q\delta^0 = (v_0'(L), v_0(s_0), \dots, v_0(s_J), v_0'(R))^T \quad (19)$$

In Equation (19), Q consists of a left endpoint derivative row $(-g, 0, g)$ on coefficients $(\delta_{-1}, \delta_0, \delta_1)$, followed by the $J+1$ interpolation rows and a right endpoint derivative row $(-g, 0, g)$ on coefficients $(\delta_{J-1}, \delta_J, \delta_{J+1})$. The derivative signs agree with Equation (11). The initial and boundary functions must satisfy the compatibility conditions given in Section 2.2.

Computational implementation

At each time level, the L_1 convolution weights are used to assemble the history vector H^n from the previously computed nodal solutions. The forcing and prescribed Dirichlet values are then incorporated into the right-hand-side vector $b^{(n+1)}$, and the banded $(J+3)$ dimensional linear system in Equation (18) is solved for the new spline coefficient vector. The corresponding nodal solution is recovered from Equation (10). This procedure consistently couples the implicit spatial collocation equations with the nonlocal temporal-memory term and provides a reproducible computational sequence for the selected mesh and model parameters.

Stability and Convergence Analysis

Stability assumptions

Consider the homogeneous perturbation recurrence associated with the linear constant-coefficient collocation system on a uniform spatial grid. Let ϵ_j^n denote the nodal error at grid point s_j and time level θ_n . Fourier analysis examines the propagation of individual spatial modes and provides a sufficient modal nonamplification criterion. The analysis assumes positive diffusion D , a fractional order $0 < \alpha < 1$, positive spline nodal symbol $S(\xi)$, and compatible homogeneous perturbation conditions. Boundary effects are considered separately from the periodic or infinite-grid Fourier symbol.

Discrete stability analysis

For a Fourier mode with real wave number ξ and imaginary unit i , the perturbation takes the form

$$\epsilon_j^n = \zeta_n \exp(i\xi j \Delta s), \quad i^2 = -1 \quad (20)$$

The symbols associated with the nodal-value, first-derivative and second-derivative spline stencils of Section 3 are, respectively,

$$S(\xi) = b + 2a \cos(\xi \Delta s) \quad (21)$$

$$P(\xi) = 2ig \sin(\xi \Delta s), \quad Q(\xi) = e + 2d \cos(\xi \Delta s) \quad (22)$$

The sign of $P(\xi)$ follows the corrected first-derivative stencil in Equation (11). Substitution into the homogeneous discrete recurrence yields

$$[\mu S(\xi) + cP(\xi) - DQ(\xi)] \zeta_{n+1} = \mu S(\xi) [\zeta_n - \Sigma_{r=1}^n w_r (\zeta_{n+1-r} - \zeta_{n-r})] \quad (23)$$

Introducing the modal denominator gives the equivalent recurrence

$$\zeta_{n+1} = [\zeta_n - \Sigma_{r=1}^n w_r (\zeta_{n+1-r} - \zeta_{n-r})] / W(\xi), \quad W(\xi) = 1 + [cP(\xi) - DQ(\xi)] / [\mu S(\xi)] \quad (24)$$

For κ in the admissible interval, $S(\xi)$ must remain positive, and the denominator cannot vanish. Since $P(\xi)$ is purely imaginary and $Q(\xi) \leq 0$ when $d \geq 0$ and $e = -2d$, the real part of $W(\xi)$ is at least one and hence $|W(\xi)| \geq 1$. The decreasing L1 weights obey $w_0 = 1$, $w_r > 0$ and $w_r > w_{r+1}$. Rearranging the history term into a convex combination of preceding modal amplitudes then gives $|\zeta_{n+1}| \leq \max_{0 \leq k \leq n} |\zeta_k|$ and, by induction, $|\zeta_n| \leq |\zeta_0|$. Thus the scheme satisfies a Fourier-mode nonamplification condition for the stated coefficients and admissible symbols. This assertion concerns modal stability and does not independently establish stability for every boundary closure or variable-coefficient extension.

Consistency and truncation error

For solutions with sufficiently regular temporal derivatives, the local L1 approximation of the Caputo operator satisfies the consistency estimate

$$|R\theta^n| \leq C\theta(\Delta\theta)^{(2-\alpha)} \quad (25)$$

For a sufficiently smooth function and an extended cubic B-spline interpolant $I_h v$ with admissible κ , the spatial interpolation estimates take the form

$$\|\partial_s^m (v - I_h v)\| \leq C_m (\Delta s)^{(4-m)}, \quad m = 0, 1, 2 \quad (26)$$

The constants depend on the regularity of the solution and the interpolation setting, rather than on the grid sizes. These interpolation inequalities provide consistency ingredients; control of the collocation coefficient error is additionally required for a fully discrete global convergence result.

Convergence estimate

At each time level, the approximation error may be decomposed into an interpolation component and a collocation component:

$$\|v - V\|_\infty \leq \|v - I_h v\|_\infty + \|I_h v - V\|_\infty \quad (27)$$

Under sufficient temporal and spatial regularity, uniform bounds for the discrete evolution operator, consistent boundary treatment and a second-order bound for the spline collocation error, the combined estimate is

$$\|v(\cdot, \theta_n) - V(\cdot, \theta_n)\|_\infty \leq C_s (\Delta s)^2 + C_t (\Delta\theta)^{(2-\alpha)} \quad (28)$$

The estimate expresses the anticipated second-order spatial contribution and L1 temporal contribution under the stated assumptions. The observed orders in Section 5 are computed directly from the available spatial-refinement data and are interpreted separately from this theoretical bound (Mohyud-Din et al., 2018).

NUMERICAL EXPERIMENTS AND RESULTS

Benchmark problems

Benchmark I considers $c = 1$, $D = 2$, $F = 0$ and the spatial interval $[0, 1]$, with initial function $v_0(s) = \exp(s)$. Its analytical solution, satisfying the governing equation because $c - D = -1$, is

$$v(s, \theta) = \exp(s) E\alpha(\theta^\alpha), \quad E\alpha(z) = \Sigma_{m=0}^\infty z^m / \Gamma(\alpha m + 1) \quad (29)$$

Benchmark II uses $c = 0$, $D = 1$, zero initial and homogeneous Dirichlet boundary values. Its analytical solution and the corresponding forcing are

$$v(s, \theta) = \theta^2 \sin(2\pi s) \quad (30)$$

$$F(s, \theta) = [2\theta^\alpha(2-\alpha)/\Gamma(3-\alpha) + 4\pi^2\theta^2] \sin(2\pi s) \quad (31)$$

The benchmark functions are evaluated using the fractional order, time step, mesh spacing and spline extension parameter specified with each table. Distinct benchmark settings are not combined into a single convergence sequence.

Verification against analytical solutions

The maximum nodal error and mesh-weighted discrete Euclidean error at final time Θ are defined as

$$\|e\|_\infty = \max_{0 \leq j \leq J} |v(s_j, \theta) - V(s_j, \theta)|, \quad \|e\|_{2,h} = [\Delta s \Sigma_{j=0}^J |v(s_j, \theta) - V(s_j, \theta)|^2]^{(1/2)} \quad (32)$$

Table 1: Benchmark I: analytical and spline-approximation values at $\Theta = 1$ ($J = 100$, $\alpha = 0.5$, $\kappa = 0.00001$, $\Delta\theta = 0.001$)

S	Analytical value	Spline approximation	Absolute error
0.1	5.53577911	5.53577900	$1.1363e-07$
0.2	6.11798209	6.11798186	$2.2948e-07$
0.3	6.76141588	6.76141555	$3.3011e-07$
0.4	7.47252019	7.47251979	$4.0100e-07$
0.5	8.25841200	8.25841157	$4.3141e-07$
0.6	9.12695678	9.12695636	$4.1533e-07$
0.7	10.08684720	10.08684680	$3.5263e-07$
0.8	11.14769020	11.14769000	$2.5034e-07$
0.9	12.32010300	12.32010290	$1.2413e-07$

The reported absolute-error column contains more significant digits than are visible in the solution-value columns; consequently, small differences between the rounded

displayed values and the errors reflect retained computational precision.

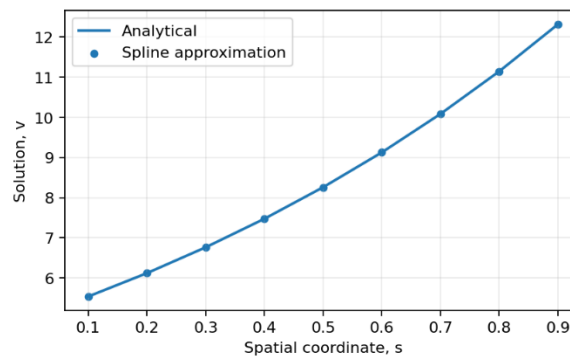


Figure 1: Analytical and spline-approximation solution profiles for Benchmark I at $\Theta = 1$

The two solution profiles agree closely at the stated spatial sample points.

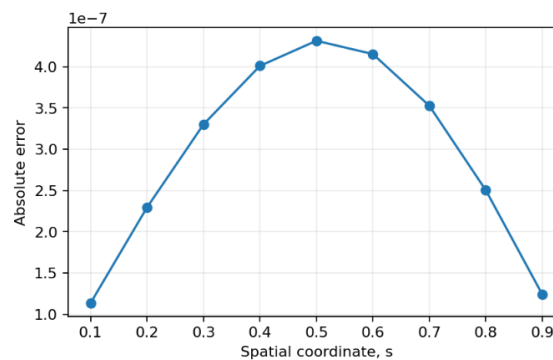


Figure 2: Pointwise absolute errors for Benchmark I at $\Theta = 1$

The pointwise absolute errors remain small and attain their largest listed value at $s = 0.5$.

Table 2: Benchmark II: pointwise values at $\Theta = 1$ ($J = 100$, $\alpha = 0.5$, $\kappa = -0.00065$, $\Delta t = 0.01$)

S	Analytical value	Spline approximation	Absolute error
0.1	0.58778525	0.58778972	$4.4636e-06$
0.2	0.95105652	0.95106374	$7.2223e-06$
0.3	0.95105652	0.95106374	$7.2223e-06$
0.4	0.58778525	0.58778972	$4.4636e-06$
0.5	0.00000000	$2.96897923e-14$	$2.9690e-14$
0.6	-0.58778530	-0.58778970	$4.4636e-06$
0.7	-0.95105650	-0.95106370	$7.2223e-06$
0.8	-0.95105650	-0.95106370	$7.2223e-06$
0.9	-0.58778530	-0.58778970	$4.4636e-06$

Table 3: Benchmark II: pointwise values at $\Theta = 10$ ($J = 50$, $\alpha = 0.3$, $\kappa = -0.0026305$, $\Delta t = 0.1$)

S	Analytical value	Spline approximation	Absolute error
0.1	58.7785252	58.7785251	$1.2446e-07$
0.2	95.1056516	95.1056514	$2.0138e-07$
0.3	95.1056516	95.1056514	$2.0138e-07$
0.4	58.7785252	58.7785251	$1.2446e-07$
0.5	0.00000000	$-5.8308913e-13$	$5.8309e-13$
0.6	-58.7785250	-58.7785250	$1.2446e-07$
0.7	-95.1056520	-95.1056510	$2.0138e-07$
0.8	-95.1056520	-95.1056510	$2.0138e-07$
0.9	-58.7785250	-58.7785250	$1.2446e-07$

Data verification note: In the supplied Benchmark II tables, the approximation at $s = 0.5$ is printed as 2.96897923 and -5.8308913 , while the corresponding errors are of orders 10^{-14} and 10^{-13} . The table entries above interpret these as $2.96897923 \times 10^{-14}$ and $-5.8308913 \times 10^{-13}$, respectively,

solely to reconcile the entries with the stated errors and analytical solution. These exponent reconstructions must be confirmed from the computational output before final submission. They are not independently verified numerical results.

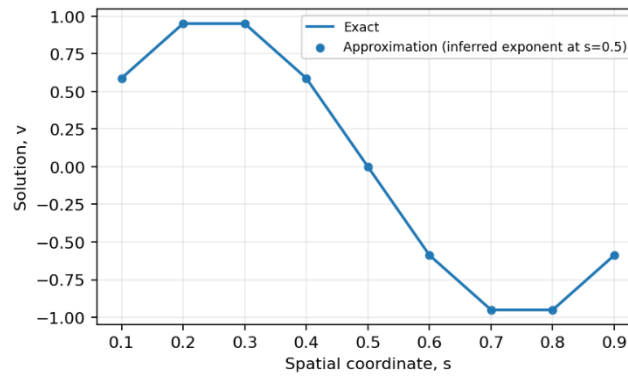


Figure 3: Analytical and spline-approximation solution profiles for Benchmark II at $\Theta = 1$; the central plotted approximation is conditional on the exponent reconstruction stated above.

Spatial convergence

For successive spatial grids related by a refinement factor of two, the observed order is evaluated from the spline maximum-error norm as

$$pobs = \log(E^\infty, J / E^\infty, 2J) / \log(2) \quad (33)$$

The following tables provide the spatial-refinement results for Benchmark I at $\Theta = 1$ and $\Delta\theta = 0.01$. The observed orders are recalculated using Equation (33), rather than carried over from the pre-existing order column. CPU times refer only to the extended-spline implementation.

Table 4: Benchmark I: spatial-refinement errors and spline execution times for $\alpha = 0.2$ ($\Theta = 1$, $\Delta\theta = 0.01$)

J	MCTB max	MCTB L_2	Spline max	Spline L_2	pobs	CPU (s)
8	0.014902	0.010412	0.00070982	0.00052421	—	0.09360
16	0.0038827	0.0026898	$6.9478e-05$	$5.0417e-05$	3.35283	0.14040
32	0.0010156	0.00066522	$3.456e-05$	$2.5203e-05$	1.00745	0.26520
64	0.0002572	0.00014842	$1.741e-06$	$1.2739e-06$	4.31112	0.73321
128	$6.3504e-05$	$2.2129e-05$	$3.8083e-07$	$1.986e-07$	2.19270	2.07481

Table 5: Benchmark I: spatial-refinement errors and spline execution times for $\alpha = 0.5$ ($\Theta = 1$, $\Delta\theta = 0.01$)

J	MCTB max	MCTB L_2	Spline max	Spline L_2	pobs	CPU (s)
8	0.0063092	0.0044047	0.00019311	0.00014246	—	0.06240
16	0.0016452	0.0011394	$7.0386e-05$	$5.1038e-05$	1.45606	0.10920
32	0.00043121	0.00028317	$2.6417e-05$	$1.9079e-05$	1.41382	0.23400
64	0.00010956	$6.4521e-05$	$5.4923e-06$	$3.8494e-06$	2.26598	0.63960
128	$2.7227e-05$	$1.0443e-05$	$5.7211e-07$	$3.0277e-07$	3.26305	2.01241

Table 6: Benchmark I: spatial-refinement errors and spline execution times for $\alpha = 0.8$ ($\Theta = 1$, $\Delta\theta = 0.01$)

J	MCTB max	MCTB L_2	Spline max	Spline L_2	Pobs	CPU (s)
8	0.0041559	0.0029052	$2.6864e-05$	0.00014246	—	0.07800
16	0.0010852	0.00075335	$1.0356e-05$	$7.4698e-06$	1.37521	0.14040
32	0.00028491	0.00018911	$1.1811e-06$	$7.866e-07$	3.13226	0.24960
64	$7.2683e-05$	$4.4967e-05$	$5.3813e-07$	$3.2574e-07$	1.13410	0.65520
128	$1.822e-05$	$8.7572e-06$	$2.4165e-07$	$1.5278e-07$	1.15504	2.01241

The observed spatial orders vary across refinement levels and fractional orders, consistent with the influence of spatial approximation, the fixed time step and numerical precision. For $\alpha = 0.8$, $J = 8$, the supplied spline L_2 value exceeds the

displayed spline maximum error by a factor inconsistent with the mesh-weighted norm in Equation (32) on $[0, 1]$. This cell is preserved as supplied and should be checked against the underlying data before publication.

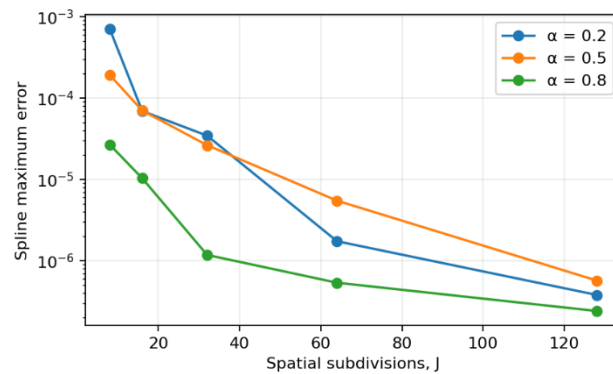


Figure 4: Spatial-refinement behaviour of the extended-spline maximum error for three fractional orders

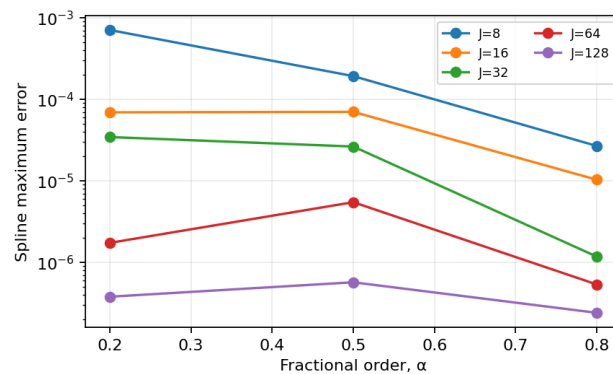


Figure 5: Dependence of the extended-spline maximum error on fractional order at fixed spatial resolutions

Temporal convergence

For solutions satisfying the regularity conditions of the L1 discretization, the temporal truncation error has order $O((\Delta\theta)^{(2-\alpha)})$. Coupled with second-order spatial approximation and the discrete stability assumptions in Section 4, this leads to the combined conditional estimate in Equation (28) (Mohyud-Din et al., 2018). The numerical data in Tables 4–6 vary spatial resolution while keeping $\Delta\theta$ fixed. They therefore evaluate spatial-refinement behaviour, whereas a separate sequence of decreasing temporal steps is needed to measure the observed temporal convergence order.

Extended B-spline parameter sensitivity and comparative performance

The spline extension parameter κ modifies the nodal coefficients and the second-derivative stencil, with $\kappa = 0$ corresponding to the conventional cubic B-spline. The benchmark settings use selected values of κ ; their results illustrate solution accuracy at these parameters. A systematic parameter-sensitivity assessment would require holding all other discretization inputs fixed while varying κ over its admissible interval.

Table 7: Benchmark II: maximum and discrete L_2 errors across CBSCM, MCTB-DQM and extended-spline methods at $\Theta = 1$, $\Delta\theta = 0.00125$ and $\alpha = 0.3$

J	CBSCM max	CBSCM L_2	MCTB max	MCTB L_2	Spline max	Spline L_2	pobs	CPU (s)
8	0.048273	0.034134	0.015762	0.00943	2.2761e-05	5.6903e-06	-	6.08404
16	0.012351	0.0087334	0.002167	0.0011924	7.4956e-06	1.3251e-06	1.60245	8.01845
32	0.0031048	0.0021955	0.00028541	0.0001504	1.7463e-06	2.1829e-07	2.10174	15.31930
64	0.00077721	0.00054957	3.6701e-05	1.8925e-05	1.3761e-07	1.2163e-08	3.66564	34.53860
128	0.0001943	0.00013739	4.6559e-06	2.3752e-06	2.2313e-08	1.3945e-09	2.62463	93.49140

For the grid sizes considered, the spline method has the smallest displayed maximum error. The reported CPU times apply to the spline method only, and are therefore used to

describe its computational scaling with J rather than as a direct cross-method timing comparison.

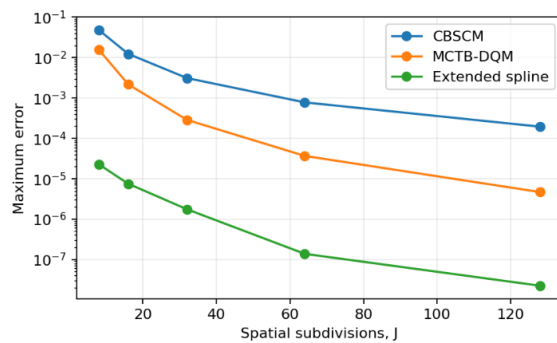


Figure 6: Maximum-error comparison for three numerical methods in Benchmark II.

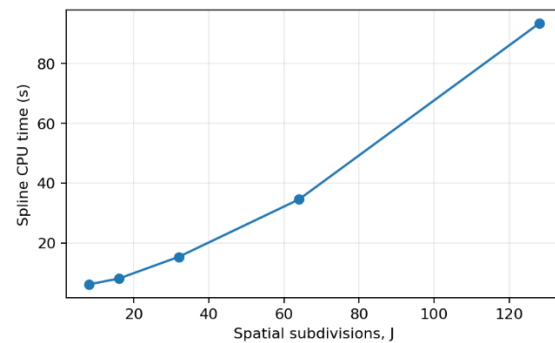


Figure 7: Extended-spline computational execution time against spatial subdivisions in Benchmark II

Taken together, the tables and figures provide complementary evidence concerning agreement with exact solutions, refinement behaviour and comparative error magnitudes under the stated benchmark configurations. Reported anomalies and inferred scientific-notation exponents are explicitly identified to preserve traceability of the numerical record.

Discussion

The numerical results provide a benchmark-based assessment of the fully implicit extended cubic B-spline collocation method for one-dimensional time-fractional advection–diffusion equations. The formulation combines the L_1 approximations of the Caputo derivative with a locally supported spline representation of the spatial operators, following the established scheme of Mohyud-Din et al. The two analytical test problems provide complementary measures of numerical accuracy through maximum absolute errors, discrete L_2 errors, and spatial convergence behaviour. For Test Problem I, the error norms demonstrate the dependence of numerical accuracy on the fractional order and spatial resolution. The fractional order γ controls the temporal memory contribution, whereas mesh refinement affects the spatial approximation. The results therefore require interpretation in terms of both discretization parameters rather than attributing changes in error exclusively to the fractional derivative. Test Problem II provides additional validation through comparisons between the analytical and numerical solution profiles. The reported agreement supports the accuracy of the formulation for the selected benchmark conditions, while the spatial refinement results provide evidence of convergence.

The stability analysis indicates nonamplification of Fourier perturbation modes under the assumptions of a uniform spatial grid, constant coefficients, and an admissible spline parameter. Nevertheless, Similarly, the expected combined error behaviour, $O(h^2 + \tau^{2-\gamma})$, depends on sufficient solution regularity and an appropriate stability estimate. The extended spline parameter λ influences the spatial derivative representation and may affect numerical accuracy. Likewise, the reported spatial convergence results cannot independently verify temporal convergence without computations at multiple time-step sizes. The execution times reported for the extended spline scheme also do not support a direct computational efficiency comparison because corresponding timing measurements for the alternative method are unavailable.

Overall, the results support the applicability of the established fully implicit extended cubic B-spline formulation to the two benchmark problems considered. Its compact spatial representation and implicit treatment of fractional memory

provide a structured numerical approach. However, the present evidence is principally a verification of published benchmark behaviour, rather than an independent demonstration of improved accuracy or computational efficiency over existing methods.

CONCLUSION

This study examined a fully implicit extended cubic B-spline collocation formulation for time-fractional advection–diffusion equations. The Caputo derivative was discretized using the L_1 approximation, while the spatial derivatives were represented through extended cubic B-spline basis functions. The mathematical assessment considered Fourier-mode stability and the expected convergence behaviour, and numerical validation was based on two analytical benchmark problems.

The reported numerical results demonstrate agreement between analytical and approximate solutions and provide evidence of spatial convergence under the selected discretization conditions. The stability analysis supports modal nonamplification within its stated assumptions. However, the available results do not independently establish temporal convergence, systematic spline-parameter sensitivity, or comparative computational efficiency. Accordingly, the study supports the numerical consistency of the established formulation for the examined benchmark cases without claiming a new discretization method or superiority over alternative schemes.

RECOMMENDATIONS

- Future research should investigate temporal convergence through systematic time-step refinement and assess its agreement with the theoretical order $O(\tau^{2-\gamma})$.
- The influence of the spline extension parameter λ on numerical accuracy and stability should be examined under controlled discretization conditions.
- Comparative computational studies using standardized benchmark problems, error norms, and execution-time measurements are recommended to evaluate the method's accuracy, efficiency, and applicability to more complex fractional transport equations.

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