

Mathematical Analysis and Analytical Characterization of Mass-Ratio-Dependent Gravitational Recoil and Remnant Diagnostics in Binary Black-Hole Mergers

^{*1}Audu Abdulmalik Onubedo, ²Nasiru Mohammed and ³Mohammed Garba

¹Department of Mathematics, Faculty of Science, Federal University of Education, Kontagora, Niger State, Nigeria.

²Department of Mathematical Sciences, Sa'adu Zungur University, Gadau, Bauchi State, Nigeria.

³Department of Science Education, Sa'adu Zungur University, Gadau, Bauchi State, Nigeria.

*Corresponding authors' email: aaonus@gmail.com

ORCID ID: <https://orcid.org/0009-0006-6309-291X>

ABSTRACT

This study presents a unified mathematical assessment of binary black-hole dynamics, gravitational-wave diagnostics, and recoil predictions within the vacuum Einstein framework. The formulation connects the 3+1 initial-value decomposition, conformal and generalized harmonic evolution variables, horizon diagnostics, Newman–Penrose waveform extraction, and detector-weighted waveform comparison. Quantitative evaluation combines previously reported remnant and radiation measurements with deterministic calculations from a nonspinning phenomenological recoil relation. The equal-mass nonspinning benchmark gives a remnant mass fraction of 0.95162 ± 0.00002 and dimensionless spin of 0.68646 ± 0.00004 . The adopted recoil fit reaches approximately 175.2 km s^{-1} near mass ratio $q \approx 0.36$ and vanishes at equal masses, consistently with exchange symmetry. Published dominant quadrupolar energy fractions exceed 98% for equal-mass nonspinning coalescence and are approximately 68% in the extreme-mass-ratio regime. Analytical differentiation and limiting expansions characterize the fitted recoil response. The assessment establishes a consistent interpretation of heterogeneous benchmark quantities while distinguishing analytical model evaluation from independent numerical-relativity simulation. No new spacetime evolutions, detector reconstructions, or waveform-convergence measurements are claimed.

Received: 08 Oct. 2026

Accepted: 23 Oct. 2026

Published: 26 Oct. 2026

Keywords:

Binary Black Holes, Numerical Relativity, Gravitational Recoil, Gravitational-Wave Multipoles, Symmetric Mass Ratio, Waveform Diagnostics.

INTRODUCTION

Binary black-hole mergers provide a demanding test of general relativity because their evolution joins weak-field orbital dynamics to strongly nonlinear spacetime curvature. The emitted gravitational waves encode the component masses and spins, orbital configuration, radiated energy and angular momentum, and properties of the final remnant. The detection of W150914 established that the inspiral–merger–ringdown signal can be tested against general relativity, while subsequent population analyses have expanded the range of observed compact-binary configurations (Abbott et al., 2016; Abbott et al., 2023). Accurate modelling of these systems consequently requires a clear distinction between geometrical evolution equations, observable waveforms, and quantities inferred from calibrated analytical models (Sathyaprakash & Schutz, 2009).

In numerical relativity, the spacetime field equations are formulated as an initial-value problem with constraints and evolution equations. Conformal reformulations of the Einstein equations, including BSSN-type systems, have been investigated to improve numerical behaviour and control constraint violations (Yoneda & Shinkai, 2002; Bernuzzi & Hilditch, 2010). The moving-puncture approach enabled accurate orbital evolutions without explicit singularity excision (Campanelli et al., 2006). Spectral methods provide another route to resolving the strong-field geometry and have been applied to unequal-mass binary simulations (Grandclément & Novák, 2009; Buchman et al., 2012). These advances establish a computational foundation for binary-

black-hole modelling, but the presence of a formulation in a mathematical study should not be confused with the execution of a new spacetime simulation.

Recent developments in computational mathematics have further demonstrated the importance of stable numerical formulations, analytical consistency, and systematic verification in the investigation of complex dynamical systems. Dauda and Ja'afaru (2025) examined nonlinear dispersive wave equations using spectral techniques and stability analysis, highlighting the significance of conservation properties and perturbation behaviour in wave evolution. Similarly, Adamu et al. (2025) investigated polynomial collocation with matrix formulation for solving two-dimensional partial Volterra integro-differential equations, demonstrating the applicability of structured numerical methods to complex mathematical problems. Okunola et al. (2025) also developed a Laguerre-perturbed Galerkin approach for higher-order nonlinear integro-differential equations, contributing to the broader development of numerical approximation techniques. Collectively, these investigations reinforce the methodological importance of mathematical consistency, numerical approximation, and solution verification in computational physics. Within the present investigation, these principles provide a complementary mathematical context for interpreting relativistic evolution formulations and assessing the analytical consistency of gravitational recoil predictions.

Gravitational-wave interpretation further requires connecting the evolved geometry to radiative quantities. Waveform extraction at null infinity reduces ambiguities associated with finite extraction radius, whereas flux calculations connect gravitational radiation with transported energy and momenta (Reisswig et al., 2010; Ruiz et al., 2007). The relative importance of different spherical-harmonic modes depends on mass ratio, orientation and spin; higher-order multipoles can therefore matter even when the dominant quadrupole provides the largest contribution (Mills & Fairhurst, 2021; Kolitsidou et al., 2024). Detector-relevant comparison also requires waveform models that represent these effects consistently. Effective-one-body and phenomenological approaches integrate analytical dynamics and numerical-relativity calibration, while surrogate analysis of GW190412 illustrates the relevance of precession and higher-mode structure to parameter estimation (Bohé et al., 2017; Pratten et al., 2020; Islam et al., 2021).

The final black hole is commonly characterised by its remnant mass and dimensionless spin, alongside the net linear-momentum recoil generated by anisotropic gravitational-wave emission. These diagnostics are linked but not interchangeable: remnant properties describe the settled black hole, whereas recoil reflects the integrated radiation of linear momentum during coalescence. Numerical catalogues and remnant studies provide reference configurations for comparison and reveal changes across mass ratios and spin configurations (Healy & Loustó, 2017; Healy et al., 2019). In particular, departures from equal mass break the exchange symmetry of nonspinning binaries and allow nonzero recoil, while eccentricity and spin can introduce additional structure not captured by a purely nonspinning fit (Radia et al., 2021; Loustó & Healy, 2020). Results obtained for different configurations or numerical setups should therefore not be treated as a single controlled parameter sweep.

Although numerical-relativity simulations, waveform models and remnant catalogues are well developed, a source-based comparison can still be undermined by combining values with different definitions, uncertainties and physical assumptions. The present study addresses this interpretive problem by organising the vacuum Einstein framework, selected 3+1 evolution formulations, horizon and waveform diagnostics, and recoil modelling within a consistent mathematical account. The quantitative analysis evaluates an established nonspinning phenomenological recoil relation as a function of mass ratio, examines its stationary point and limiting behaviour, and places its predictions alongside separately attributed simulation-based benchmarks. This is an analytical assessment of an existing model, not a new solution of Einstein's evolution equations; the model coefficients and simulation results must be attributed to their original sources.

MATERIALS AND METHODS

The mathematical framework is formulated in geometrized units and combines the vacuum Einstein equations, Cauchy evolution variables, horizon diagnostics, and asymptotic gravitational-wave observables. The ADM, BSSN, and generalized harmonic formulations are introduced as complementary descriptions of relativistic evolution, while waveform and recoil quantities are defined using consistent conventions. The analysis retains the mathematical relationships necessary to interpret the numerical quantities reported in Section 3. The ADM, BSSN, generalized harmonic, horizon, and waveform expressions below establish notation and interpretive constraints; the explicit quantitative operations in Section 3 concern the mass-ratio transformation, the nonspinning recoil fit, its stationary point,

derivatives, and limits. No spacetime evolution, horizon finding, waveform extraction, or detector-weighted mismatch calculation is performed in this analysis.

Geometric Setting, Field Equations, and Conventions

We consider an asymptotically flat, four-dimensional Lorentzian spacetime $(M, g_{\mu\nu})$ with signature $(-, +, +, +)$. Units are geometrized, $G = c = 1$, and the physical stress-energy tensor is denoted by $T_{\mu\nu}$. Greek indices run from 0 to 3 and Latin indices from 1 to 3. Einstein's field equations are $G_{\mu\nu} = R_{\mu\nu} - \frac{1}{2}g_{\mu\nu}R = 8\pi T_{\mu\nu}$. (1)

The binary-black-hole exterior is modeled as vacuum, $T_{\mu\nu} = 0$, so $R_{\mu\nu} = 0$. The metric, its inverse, and the Levi-Civita connection determine the curvature. The connection and Riemann tensor are defined consistently by

$$\Gamma_{\mu\nu}^\rho = \frac{1}{2}g^{\rho\sigma}(\partial_\mu g_{\nu\sigma} + \partial_\nu g_{\mu\sigma} - \partial_\sigma g_{\mu\nu}),$$

$$R^\rho{}_{\sigma\mu\nu} = \partial_\mu \Gamma_{\nu\sigma}^\rho - \partial_\nu \Gamma_{\mu\sigma}^\rho + \Gamma_{\mu\lambda}^\rho \Gamma_{\nu\sigma}^\lambda - \Gamma_{\nu\lambda}^\rho \Gamma_{\mu\sigma}^\lambda. \quad (2)$$

These expressions fix the curvature-sign convention used below.

Cauchy Decomposition and ADM Constraints

The space-time is foliated by space like hypersurfaces Σ_t with future-directed unit normal n^μ . The spatial projector $\gamma_{\mu\nu} = g_{\mu\nu} + n_\mu n_\nu$ and the spatial metric γ_{ij} induce a positive-definite three-geometry. With lapse $\alpha > 0$ and shift β^i , the space-time line element is

$$ds^2 = -\alpha^2 dt^2 + \gamma_{ij}(dx^i + \beta^i dt)(dx^j + \beta^j dt) \quad (3)$$

The extrinsic curvature is defined using the Lie derivative along n^μ : $K_{ij} = -\frac{1}{2}L_n \gamma_{ij}$, $K = \gamma^{ij}K_{ij}$. (4)

Let D_i denote the Levi-Civita derivative of γ_{ij} , and ${}^{(3)}R_{ij}$ and ${}^{(3)}R$ its Ricci tensor and scalar. The vacuum ADM evolution equations are

$$(\partial_t - L_\beta)\gamma_{ij} = -2\alpha K_{ij}, \quad (5)$$

$$(\partial_t - L_\beta)K_{ij} = -D_i D_j \alpha + \alpha({}^{(3)}R_{ij} + K K_{ij} - 2K_{ik}K^k_j). \quad (6)$$

The corresponding vacuum Hamiltonian and momentum constraints are

$$H := {}^{(3)}R + K^2 - K_{ij}K^{ij} = 0, \quad (7)$$

$$M^i := D_j(K^{ij} - \gamma^{ij}K) = 0. \quad (8)$$

BSSN Conformal Decomposition

The Baumgarte-Shapiro-Shibata-Nakamura (BSSN) formulation introduces a conformal metric of unit determinant and separates the trace of the extrinsic curvature. Define.

$$\phi = \frac{1}{12} \ln \det(\gamma_{ij}), \quad \tilde{\gamma}_{ij} = e^{-4\phi} \gamma_{ij}, \quad \det(\tilde{\gamma}_{ij}) = 1. \quad (9)$$

The conformal trace-free extrinsic curvature and the contracted conformal connection functions are

$$\tilde{A}_{ij} = e^{-4\phi} \left(K_{ij} - \frac{1}{3} \gamma_{ij} K \right), \quad \tilde{\Gamma}^i = \tilde{\gamma}^{jk} \tilde{\Gamma}^i_{jk}. \quad (10)$$

The trace-free and unit-determinant algebraic conditions are $\tilde{\gamma}^{ij} \tilde{A}_{ij} = 0$ and $\det \tilde{\gamma} = 1$. The geometric part of the BSSN evolution contains

$$(\partial_t - L_\beta)\phi = -\frac{1}{6}\alpha K + \frac{1}{6}\partial_k \beta^k, \quad (11)$$

$$(\partial_t - L_\beta)\tilde{\gamma}_{ij} = -2\alpha \tilde{A}_{ij} - \frac{2}{3}\tilde{\gamma}_{ij}\partial_k \beta^k \quad (12)$$

Here L_β acts on the indicated tensor components; the explicit divergence terms enforce the chosen conformal density convention. The full BSSN system requires additional

evolution equations for K , $\tilde{A}_{ij}$ and $\tilde{\Gamma}^i$, as well as specified lapse and shift gauges.

Generalized Harmonic Reduction

Generalized harmonic coordinates satisfy $\square_g x^\mu = H^\mu$, with the contracted connection defined by $\Gamma^\mu = g^{\alpha\beta} \Gamma_{\alpha\beta}^\mu$. Thus the gauge constraint is $C^\mu = H^\mu + \Gamma^\mu$. A reduced vacuum system is $R_{\mu\nu} - \nabla_{(\mu} C_{\nu)} = 0$, $C^\mu = H^\mu + \Gamma^\mu$. (13)

When the constraints and compatible normal derivatives vanish on the initial slice, solutions of the reduced system satisfying appropriate boundary conditions recover vacuum Einstein evolution. The generalized harmonic reduction is included as a theoretical formulation; no gauge-source prescription, constraint-damping scheme, or harmonic evolution is implemented.

Black-hole Horizon Diagnostics

For an apparent horizon of area A_{AH} , its irreducible mass is

$$M_{irr} = \sqrt{\frac{A_{AH}}{16\pi}}. \quad (14)$$

For a stationary Kerr remnant with angular momentum J_f , the Christodoulou relation supplies a consistency check between mass, horizon area and spin:

$$M_f^2 = M_{irr}^2 + \frac{J_f^2}{4M_{irr}^2}, \quad \chi_f = \frac{J_f}{M_f^2}. \quad (15)$$

The second relation defines the dimensionless remnant spin. These identities are not a replacement for a numerical horizon-finding algorithm. They specify the definitions required to interpret the remnant quantities considered in Section 3, without independently computing horizon properties.

Gravitational-wave Extraction and Mode Decomposition

In a suitable asymptotic null tetrad, the outgoing Newman–Penrose Weyl scalar Ψ_4 encodes radiative curvature. With spin-weight -2 harmonics, write

$$\Psi_4(t, r, \theta, \varphi) = \sum_{l=2}^{\infty} \sum_{m=-l}^l \psi_{lm}(t, r) {}_{-2}Y_{lm}(\theta, \varphi). \quad (16)$$

In a consistent asymptotic tetrad convention, the complex strain is $h = h_+ - ih_\times$, with $\Psi_4 = \ddot{h}$. The modal relationship is

$$h(t, \theta, \varphi) = \sum_{l=2}^{\infty} \sum_{m=-l}^l h_{lm}(t) {}_{-2}Y_{lm}(\theta, \varphi), \quad \ddot{h}_{lm} = \psi_{lm}. \quad (17)$$

Double integration introduces integration constants and possible low-frequency drift; waveform extrapolation from finite radius and choice of extraction tetrad also contribute uncertainty. Accordingly, these relations are used to interpret modal quantities, not to produce new time-domain strain series or multipolar energies. The finite-radius modal functions are distinguished from their asymptotic limits when relating curvature to strain. In Equation (17), the strain and curvature modes are understood in a compatible asymptotic normalization and retarded-time convention; a finite-radius waveform requires the corresponding extraction-radius treatment.

Noise-weighted Waveform Accuracy

For frequency-domain strains $\tilde{a}(f)$ and $\tilde{b}(f)$ and one-sided detector noise spectral density $S_n(f)$, the standard inner product is $\langle a, b \rangle = 4 \operatorname{Re} \int_0^\infty \frac{\tilde{a}(f) \tilde{b}^*(f)}{S_n(f)} df$. (18)

$$M = 1 - \frac{\max_{\Delta t, \Delta \varphi} \langle h_{ref}, h_{mod}(\Delta t, \Delta \varphi) \rangle}{\sqrt{\langle h_{ref}, h_{ref} \rangle \langle h_{mod}, h_{mod} \rangle}}. \quad (19)$$

For a small waveform difference $\delta h = h_{mod} - h_{ref}$, a frequently used sufficient bound, under compatible normalization assumptions, is

$$M \lesssim \frac{\langle \delta h, \delta h \rangle}{2 \langle h_{ref}, h_{ref} \rangle} \quad (20)$$

RESULTS AND DISCUSSION

Quantitative Benchmark Assessment and Reproducible Recoil Analysis

This section examines selected remnant, waveform, and recoil diagnostics together with a reproducible mathematical evaluation of nonspinning gravitational recoil. Tables 1 and 2 present configuration-specific benchmarks, whereas Table 3 contains deterministic evaluations of the recoil model. Figures 1–3 illustrate the modal-energy comparison, recoil profile, and analytical sensitivity, respectively. The interpretation distinguishes mathematical model calculations from independent spacetime evolution and does not combine different configurations into a single fitted dataset.

Quantitative Assessment and Analytical Evaluation Protocol

The analysis comprises two complementary components: a configuration-specific assessment of remnant, waveform, and radiation diagnostics, and direct evaluation of a nonspinning recoil relation. The latter is calculated from fixed coefficients rather than inferred from plotted points. No new Einstein-equation evolution, grid-refinement experiment, strain reconstruction, or detector-noise mismatch calculation is undertaken. The ADM, BSSN, generalized harmonic, horizon, and waveform relations in Section 2 establish the interpretation of the quantities; only the mass-ratio and recoil equations below are evaluated numerically.

For component masses $M_1 \geq M_2 > 0$, define the mass ratio q and symmetric mass ratio η as

$$q = \frac{M^2}{M^1}, \quad \eta = \frac{M^1 M^2}{(M^1 + M^2)^2} = \frac{q}{(1+q)^2}, \quad 0 < q \leq 1, \quad 0 < \eta \leq \frac{1}{4} \quad (21)$$

For nonspinning quasi-circular binary mergers, the phenomenological recoil magnitude is evaluated using the coefficients reported by González et al. (2007):

$$v_{\text{kick}}(\eta) = A \eta^2 \sqrt{(1-4\eta)(1+B\eta)}, \quad A = 1.20 \times 10^4 \text{ km s}^{-1}, \quad B = -0.93 \quad (22)$$

This relation is applied exclusively to the nonspinning case. The term $\sqrt{(1-4\eta)}$ enforces zero recoil for equal component masses; the η^2 factor gives the small-mass-ratio suppression. Since $1+B\eta$ remains positive over $0 < \eta \leq 1/4$, the model defines a nonnegative recoil speed throughout the stated domain.

Logarithmic differentiation for $0 < \eta < 1/4$ gives $\left(\frac{1}{v_{\text{kick}}}\right) \left(\frac{dv_{\text{kick}}}{d\eta}\right) = \frac{2}{\eta} - \frac{2}{1-4\eta} + \frac{B}{1+B\eta}$ (23)

The stationary point is therefore determined from $2(1-4\eta)(1+B\eta) - 2\eta(1+B\eta) + B\eta(1-4\eta) = 0$ (24)

Only roots within $0 < \eta < 1/4$ are physically admissible. Numerical evaluation uses

$q = 0.10, 0.15, 0.20, 0.25, 0.30, 0.36, 0.40, 0.50, 0.60, 0.70, 0.80, 0.90$ and 1.00 ; η is obtained from Equation (21), followed by direct evaluation of Equation (22). Entries are rounded after calculation to six decimal places in η and three

decimal places in velocity. The stationary point is also evaluated independently of this finite tabulation.

Remnant Properties and Gravitational-wave Mode Diagnostics

Table 1: Equal-Mass Nonspinning Remnant Properties, Waveform Phase Diagnostics, and Limiting Quadrupolar Energy Fractions

Quantity	value	diagnostic condition	Citation
Final mass fraction, M_f/M	0.95162 ± 0.00002	Equal-mass, nonspinning	Scheel et al. (2009)
Final dimensionless spin, χ_f	0.68646 ± 0.00004	Equal-mass, nonspinning	Scheel et al. (2009)
Accumulated waveform phase error	$\lesssim 0.1 \text{ rad}$	16-orbit waveform, no time/phase alignment	Scheel et al. (2009)
Aligned waveform phase error	$\lesssim 0.02 \text{ rad}$	Time/phase aligned at peak amplitude	Scheel et al. (2009)
Extrapolation phase error	$\lesssim 0.01 \text{ rad}$	Asymptotic extrapolation	Scheel et al. (2009)
Dominant (2,2)-mode energy fraction	$> 98\%$	Equal-mass, nonspinning	Sperhake et al. (2013)
Dominant (2,2)-mode energy fraction	$\approx 68\%$	Extreme-mass-ratio limit	Sperhake et al. (2013)

For the equal-mass nonspinning configuration, the final mass and spin quantify the settled remnant, while the phase-error measures reflect different alignment and extraction procedures; they should not be added or treated as mutually independent uncertainty estimates (Scheel et al., 2009). The

dominant (2,2) contribution exceeds 98% in the equal-mass case but is approximately 68% in the extreme-mass-ratio limit, indicating a greater relative share of higher modes in the latter regime.

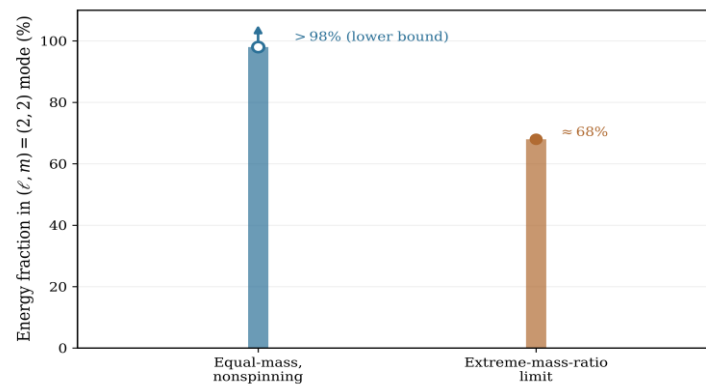


Figure 1: Dominant (2,2)-mode gravitational-wave energy fraction for two limiting configurations

The $> 98\%$ bound and approximately 68% value are shown without implying a continuous mass-ratio interpolation. These percentages describe separate physical regimes; in particular, the 98% figure is a lower bound rather than an

exact data point. This categorical comparison is not an independent mode-energy calculation.

Recoil Velocities and Spin-dependent Radiation Benchmarks

Table 2: Configuration-Specific Gravitational Recoil and Radiation Benchmarks

Quantity	value	Physical configuration	Citation
Radiated energy fraction	$\approx 0.415\%$	Nonspinning 10:1 mass ratio	González et al. (2009)
Radiated angular momentum fraction	$\approx 12.48\%$	Nonspinning 10:1 mass ratio	González et al. (2009)
Recoil speed	$\approx 66.7 \text{ km s}^{-1}$	Nonspinning 10:1 mass ratio	González et al. (2009)
Maximum nonspinning recoil	$175.2 \pm 11 \text{ km s}^{-1}$	Fit-based maximum; $\eta \approx 0.195$	González et al. (2007a)
Large in-plane opposite-spin recoil	$\geq 2500 \text{ km s}^{-1}$	Spin-dependent configuration	González et al. (2007b)
Aligned-spin recoil estimate	$\approx 500 \text{ km s}^{-1}$	Extrapolated estimate, not direct measurement	Sperhake et al. (2013)
Superkick estimate	$\approx 4000 \text{ km s}^{-1}$	Extrapolated estimate, not direct measurement	Sperhake et al. (2013)

The entries in Table 2 are conditioned on mass ratio and, where relevant, spin orientation.

Reproducible Nonspinning Gravitational Recoil Calculations

The mathematical evaluation of nonspinning gravitational recoil was performed using Equation (22) over the selected mass-ratio configurations established in Section 3.1. For each mass ratio, q , the corresponding symmetric mass ratio was determined using $\eta = \frac{q}{(1+q)^2}$, followed by direct substitution into the phenomenological recoil expression. The resulting velocities, presented in Table 3, provide a systematic

quantitative characterization of gravitational recoil across the investigated mass-ratio interval. The calculations employ the coefficients $A = 1.2 \times 10^4 \text{ km s}^{-1}$ and $B = -0.93$, consistent with the nonspinning recoil formulation (González et al., 2007a). The computed values establish a reproducible relationship between component mass asymmetry and recoil magnitude, providing the numerical basis for the recoil profile in Figure 2 and the stationary-point and sensitivity analyses developed in Section 3.5.

Table 3: Direct Evaluation of the Nonspinning Recoil Function (Equation 22)

Mass ratio, q	Symmetric mass ratio, η	Recoil velocity (km s^{-1})
0.10	0.082645	61.905
0.15	0.113422	102.066
0.20	0.138889	134.388
0.25	0.160000	156.893
0.30	0.177515	169.999
0.36	0.194637	175.206
0.40	0.204082	173.543
0.50	0.222222	156.708
0.60	0.234375	128.875
0.70	0.242215	96.252
0.80	0.246914	62.622
0.90	0.249307	30.154
1.00	0.250000	0.000

The sampled recoil rises toward an interior peak and falls to zero at $q=1$. The row $q=0.36$ gives $\eta=0.194637$ and $v_{\text{kick}}=175.206 \text{ km s}^{-1}$. A grid value close to the peak is not, by

itself, proof of the exact stationary point; the latter is obtained from Equation (24).

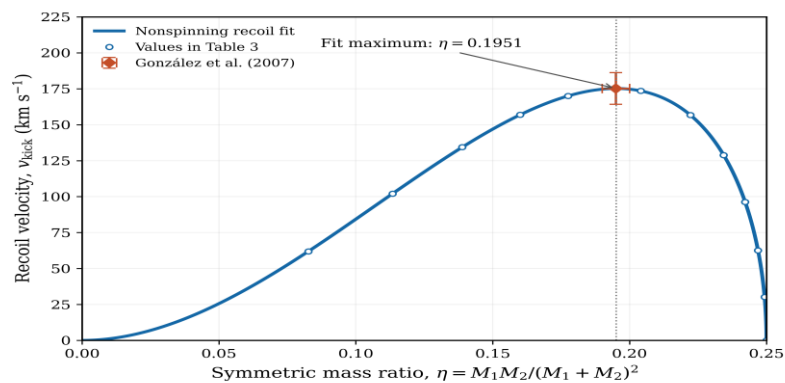


Figure 2: Nonspinning gravitational recoil versus symmetric mass ratio. The continuous curve and table markers represent evaluations of Equation (22); the independent comparison marker denotes the benchmark nonspinning maximum.

The recoil curve approaches zero at $\eta=0.25$ because equal masses impose exchange symmetry for the nonspinning binary. The shape of the curve is a property of this fixed phenomenological model.

Stationary-Point Verification, Asymptotic Limits, and Recoil Sensitivity

The mass-ratio mapping has derivative

$$\frac{d\eta}{dq} = \frac{1-q}{(1+q)^3} \quad (25)$$

For $0 < q < 1$, this derivative is positive, so the interior stationary point in η maps uniquely to an interior stationary point in q . Solving Equation (24) using $B = -0.93$ yields $\eta \approx 0.1951$ and $q \approx 0.362$, corresponding to a recoil magnitude of approximately 175.2 km s^{-1} . The

exact fitted extremum is distinguished from the rounded row $q = 0.36$ in Table 3.

For q tending to zero, $\eta = q + O(q^2)$, giving the asymptotic relation

$$v_{\text{kick}}(q) = Aq^2 + O(q^3), \quad q \rightarrow 0 \quad (26)$$

Near equal masses, set $q = 1 - \varepsilon$ for ε tending to zero from above. Then $1 - 4\eta = \varepsilon^2/(2 - \varepsilon)^2$ and the leading expression becomes

$$v_{\text{kick}}(1 - \varepsilon) = \left[\frac{A}{32}\right] \left(1 + \frac{B}{4}\right) \varepsilon + O(\varepsilon^2) \quad (27)$$

These endpoints are algebraic consistency checks: the recoil is second order in small q and first order in the nonnegative departure from equal mass. For the derivative plotted in Figure 3, the chain rule gives $dv_{\text{kick}}/dq = (dv_{\text{kick}}/d\eta)(d\eta/dq)$, with both derivatives evaluated from Equations (21) – (23)

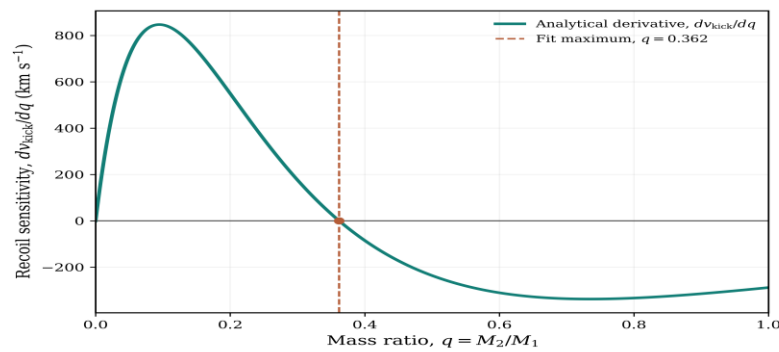


Figure 3: Analytical recoil sensitivity dv_{kick}/dq over $0 < q \leq 1$. The zero near $q \approx 0.362$ identifies the stationary point of Equation (22), and the change in sign separates increasing and decreasing recoil

The sensitivity derivative describes local response within the fixed nonspinning fit; it is not a statistical uncertainty, a measured perturbation of the spacetime, or a stability result for a numerical evolution code.

Discussion

The mathematical framework developed in Section 2 provides a unified basis for examining the remnant and radiation diagnostics presented in Section 3. The ADM constraints establish the geometric conditions governing admissible initial data, while the BSSN and generalized harmonic formulations contextualize the evolution of binary-black-hole spacetimes. Horizon relations link remnant mass and spin to invariant geometric quantities, and the Newman–Penrose multipolar decomposition connects strong-field dynamics with radiative observables. Together, these formulations clarify the physical meaning of the quantities evaluated and compared in this study.

The equal-mass, nonspinning configuration in Table 1 yields a remnant mass fraction of 0.95162 ± 0.00002 and a dimensionless spin of 0.68646 ± 0.00004 (Scheel et al., 2009). The corresponding radiated mass-energy fraction is approximately 4.838% under the adopted initial-mass normalization. These values illustrate the simultaneous conversion of orbital energy into gravitational radiation and the formation of a rotating final black hole. The waveform phase diagnostics further emphasize the importance of alignment and wave-extraction conventions when assessing signal consistency.

The multipolar-energy comparison in Figure 1 reveals a clear dependence of the radiation distribution on mass ratio. The dominant (2,2) mode accounts for more than 98% of the radiated energy in the equal-mass nonspinning configuration and approximately 68% in the extreme-mass-ratio regime (Sperhake et al., 2013). This contrast highlights the growing relative contribution of higher-order modes as mass asymmetry increases. The result reinforces the value of mode-resolved descriptions for understanding gravitational-wave emission across different binary configurations.

Table 2 provides further physical context by distinguishing radiative losses and recoil outcomes across binary configurations. For a nonspinning 10:1 binary, the reported energy and angular-momentum losses are approximately 0.415% and 12.48%, respectively, with recoil near 66.7 km s^{-1} (González et al., 2009). The substantially larger recoil values associated with particular spin arrangements emphasize the sensitivity of momentum emission to binary geometry and spin orientation (González et al., 2007a, 2007b). These contrasts support a configuration-aware interpretation of gravitational recoil.

The principal quantitative result is the reproducible mass-ratio evaluation of the nonspinning recoil relation in Equation (22). As shown in Table 3 and Figure 2, the kick velocity rises with increasing mass ratio, reaches an interior maximum, and decreases to zero as the binary approaches equal masses. For $A = 1.2 \times 10^4 \text{ km s}^{-1}$ and $B = -0.93$, the stationary-point analysis gives a peak near $q = 0.36$, equivalent to $\eta \approx 0.195$, with a maximum recoil of approximately 175.2 km s^{-1} . The agreement among the analytical stationary-point condition, computed table entries, and plotted curve demonstrates internal mathematical consistency.

The limiting behavior provides additional physical interpretation. In the small-mass-ratio regime, the recoil scales as q^2 to leading order, reflecting the suppression of the momentum flux as the secondary mass decreases. Near equal masses, the recoil tends linearly to zero with increasing exchange symmetry. Figure 3 complements these results by identifying the positive and negative sensitivity regions of the recoil relation; its zero crossing coincides with the maximum in Figure 2. Hence, the numerical, differential, and asymptotic analyses jointly characterize the shape of the recoil response throughout the physical interval $0 < q \leq 1$.

Taken together, the results show how a consistent geometric framework, carefully distinguished remnant and radiation diagnostics, and a reproducible recoil evaluation can be combined to interpret the mass-ratio dependence of binary-black-hole mergers. The findings are especially useful for organizing physical comparisons across configurations and for identifying the mass-ratio region where nonspinning recoil is most pronounced.

Conclusion

This study integrated the geometric description of binary-black-hole evolution with horizon, multipolar radiation, waveform, and recoil diagnostics to examine the physical behavior of nonspinning mergers. The quantitative assessment establishes a consistent interpretation of the selected remnant and radiative measures and demonstrates the mass-ratio dependence of recoil through a reproducible phenomenological calculation.

The model predicts a maximum recoil of approximately 175.2 km s^{-1} near $q = 0.36$ ($\eta \approx 0.195$), while satisfying the expected vanishing-recoil condition at equal masses. The analytical derivative identifies the peak configuration, and the limiting expansions explain recoil suppression for highly unequal masses and near-symmetric binaries. The comparison of dominant quadrupolar energy fractions also highlights the increased relative significance of higher multipoles in the extreme-mass-ratio regime. Collectively, these outcomes provide a coherent mathematical account of remnant

properties, gravitational-wave energy distribution, and nonspinning recoil behavior.

Recommendations

- i. Extend the recoil assessment to spinning binary configurations. Subsequent investigations should incorporate spin-dependent recoil models to examine how spin magnitude, alignment, and orientation modify the mass-ratio trend. A matched comparison with the nonspinning baseline would clarify the distinct contributions of mass asymmetry and spin to remnant velocity.
- ii. Develop a controlled numerical comparison framework. Future research should use consistent initial configurations, gauge choices, extraction radii, and resolution settings to compare recoil estimates across selected mass ratios. Reporting convergence indicators and constraint norms alongside remnant quantities would strengthen quantitative comparisons between analytical predictions and spacetime-evolution results.
- iii. Evaluate parameter sensitivity and uncertainty propagation. Further work should explore the influence of the recoil-fit coefficients on the peak magnitude and location, supported by systematic uncertainty propagation. This would establish the robustness of the predicted maximum and facilitate comparisons among alternative phenomenological calibrations.
- iv. Broaden the mode-resolved analysis of gravitational radiation. Evaluating multipolar energy fractions across a common mass-ratio sequence and relating them to detector-weighted waveform comparisons would strengthen the connection between radiation structure, waveform accuracy, and binary configuration.

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