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Transient Analysis of Series and Parallel RLC Circuits using a Higher-Order Hybrid Block Backward Differentiation Formula with Off-Step Points

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ABSTRACT
Switching and fault events excite transient currents and voltages in RLC circuits, and the governing second-order equation becomes stiff when the damping is strong. This paper applies a hybrid block backward differentiation formula (HSBBDF), with two off-step points, a stability-control parameter and step-size ratios r = 1, 2 and 5/8, to the source-free response of series and parallel RLC circuits in the under-damped, critically damped and over-damped regimes. The properties of the scheme were re-examined. It is of order six, which a constant-step convergence test on the six circuit problems supports (observed rates 5.7 to 6.3 in 17 of the 18 estimates), it is zero-stable for the three step-size ratios, and for r = 1 it is A(α)-stable with α ≈ 88.7°. A variable-step implementation with a Milne-type error estimate was run at tolerances 10^{-2}, 10^{-4} and 10^{-6} and compared with the variable-step variable-order 2-point diagonal block BDF (VSVO 2DBBDF) and MATLAB's ode15s, using the results published for those methods, and with the variable-order BDF code of SciPy run on the same platform. HSBBDF gave a smaller maximum mixed error than VSVO 2DBBDF in 17 of the 18 cases and than ode15s in 16 of 18, with 1.6 to 5.4 times fewer steps. Its Python prototype was, however, slower in wall-clock time than SciPy's BDF, and its global error exceeded the requested tolerance in some runs.

Keywords: Transient Analysis, RLC Circuits, Block Backward Differentiation Formula, Off-Step Points, Variable Step Size

INTRODUCTION

A sudden change in an electrical network, such as switching a source, interrupting a branch or a short circuit, is followed by a transient in which currents and voltages can briefly exceed their steady-state values and damage components. In a circuit with one inductor and one capacitor the transient obeys a linear second-order differential equation, and the response is under-damped, critically damped or over-damped according to the resistance (Alexander & Sadiku, 2017). Closed-form solutions are available for the source-free case, but a numerical solver is still needed as soon as the circuit is extended with nonlinear or time-varying elements, and the problem is stiff when the two characteristic roots differ widely, as in a strongly over-damped circuit.

The backward differentiation formulae (BDF) of Curtiss and Hirschfelder (1952) are the classical implicit family for stiff problems. Block versions compute several solution values at once (Ibrahim et al., 2007), (Musa & Unwala, 2019) and a super class with a free stability-control parameter was given by Suleiman et al. (2014). For second-order equations, block BDF that solve the equation directly, without reduction to a first-order system, were developed by Zainuddin et al. (2016). Most recently, Zainuddin et al. (2026) derived a variable-step variable-order 2-point diagonal block BDF (VSVO 2DBBDF) of orders two and three and tested it on series and parallel RLC circuits in all three damping regimes, comparing it with MATLAB's ode15s (Shampine & Reichelt, 1997). A lot more method methods are formulated with good accuracy can be found in Sagir & Abdullahi (2023a, 2023b), Abdullahi *et. al* (2023, 2021a,2021b). Their study reports step counts, failed steps, maximum mixed error and run time at three tolerances and provides the benchmark used here.

An implementation of a higher-order diagonally implicit with two off-step points of Sagir et al. (2022 & 2026), which was developed for damped mass–spring problems. It produces two solution points and two off-step points per block from three back values and is of order six. The present paper applies it to the six RLC circuits of Zainuddin et al. (2026), re-derives and checks its properties (order, error constants, zero-stability and absolute stability), describes a variable-step implementation, and compares it with the two methods of that study and with a variable-order BDF code run on the same machine. Section 2 describes the circuits, Section 3 the scheme and its properties, Section 4 the implementation, Section 5 the results, and Section 6 the conclusions.

Circuit Models

The source-free (natural) response of the two circuits of Zainuddin et al. (2026) was considered. In the series circuit the inductor current i_L satisfies $i_L'' + (R/L)\, i_L' + i_L/(LC) = 0$ with L = 1 H and C = 0.01 F, and the initial values $i_L(0) = 5$ A and $v_C(0) = 0$ V give $i_L'(0) = -(R\, i_L(0) + v_C(0))/L$. In the parallel circuit the capacitor voltage satisfies $v_C'' + v_C'/(RC) + v_C/(LC) = 0$ with L = 0.1 H and C = 0.001 F, with $v_C(0) = 5$ V and $i_L(0) = 0$ A, so that $v_C'(0) = -(v_C(0) + R\, i_L(0))/(RC)$. Both have the form

$$y'' + by' + cy = 0, \quad y(0) = y_0, \quad y'(0) = y_0' \tag{1}$$

with b = R/L, c = 1/(LC) for the series circuit and b = 1/(RC), c = 1/(LC) for the parallel circuit. The motion is under-damped if $b^2 < 4c$, critically damped if $b^2 = 4c$ and over-damped if $b^2 > 4c$. This corresponds to R below, equal to and above $2\sqrt{(L/C)} = 20\ \Omega$ in the series circuit, and to R above, equal to and below $\tfrac{1}{2}\sqrt{(L/C)} = 5\ \Omega$ in the parallel circuit. Writing $y_1 = y$ and $y_2 = y'$ gives the first-order system

$$y_1' = y_2, \qquad y_2' = -c\, y_1 - b\, y_2 \tag{2}$$

which is the form integrated by HSBBDF. The six cases, their characteristic roots and exact solutions are in Table 1; each

exact solution was checked symbolically against the differential equation and the initial values. Two points of terminology differ from the source paper: its case a) for the series circuit is called undamped, although R = 16 Ω gives roots −8 ± 6i and the circuit is under-damped, and the characteristic equation is $s^2 + bs + c = 0$. In the series circuit the resistor and inductor voltages are $v_R = R\, i_L$ and $v_L = L\, di_L/dt$, and the capacitor voltage follows from $v_C = -(v_R + v_L)$.

Table 1: Circuit Parameters, Characteristic Roots and Exact Solutions (L = 1 H, C = 0.01 F series; L = 0.1 H, C = 0.001 F Parallel)

Circuit and regime	R (Ω)	(b, c)	Roots	Exact solution
Series, under-damped	16	(16, 100)	−8 ± 6i	$i_L = e^{-8t}(5\cos 6t - (20/3)\sin 6t)$
Series, critically damped	20	(20, 100)	−10, −10	$i_L = 5e^{-10t} - 50te^{-10t}$
Series, over-damped	52	(52, 100)	−2, −50	$i_L = -(5/24)e^{-2t} + (125/24)e^{-50t}$
Parallel, under-damped	6.25	(160, 10^4)	−80 ± 60i	$v_C = e^{-80t}(5\cos 60t - (20/3)\sin 60t)$
Parallel, critically damped	5	(200, 10^4)	−100, −100	$v_C = 5e^{-100t} - 500te^{-100t}$
Parallel, over-damped	200/401	(2005, 10^4)	−5, −2000	$v_C = -(5/399)e^{-5t} + (2000/399)e^{-2000t}$

The Hybrid Block Formula and it's Properties

Formula and Coefficients

With step size h and $x_{n+j} = x_n + jh$, one block of HSBBDF computes $y_{n+1/2}$, y_{n+1}, $y_{n+3/2}$ and y_{n+2} from the back values y_{n-2r}, y_{n-r} and y_n. Formula i is

$$\sum_{j=0}^{6} \alpha_{j,i}\, y_{n+t_j} = h\beta_i (f_{n+k_i} - \rho f_{n+k_i-2}), \quad i = 1, 2, 3, 4 \tag{3}$$

with k_i = 1/2, 1, 3/2, 2, nodes $t_0 = -2r$, $t_1 = -r$, $t_2 = 0$, t_3 = 1/2, $t_4 = 1$, t_5 = 3/2, $t_6 = 2$, and the coefficient of y_{n+ki} equal to one. The parameter ρ is the stability-control parameter of Suleiman et al. (2014) and was set to ρ = 1/2. Because the back values lie at $x_n - rh$ and $x_n - 2rh$, r is the ratio of the previous step to the new step: r = 1 keeps the step, r = 2 halves it and r = 5/8 enlarges it by the factor 8/5. The coefficients follow from requiring the Taylor expansion of each formula about x_n to vanish up to order six, which gives seven linear conditions for the six $\alpha_{j,i}$ and β_i of each formula. They were obtained symbolically, and the coefficients for r = 1 are listed in Table 2; those for r = 2 and 5/8 come from the same conditions.

Table 2: Coefficients of HSBBDF for r = 1 and ρ = 1/2

i	$\alpha_{0,i}$	$\alpha_{1,i}$	$\alpha_{2,i}$	$\alpha_{3,i}$	$\alpha_{4,i}$	$\alpha_{5,i}$	$\alpha_{6,i}$	β_i
1	873/11872	49/1272	−2045/2544	1	−225/424	299/1113	−235/5088	10/53
2	−47/2450	−11/35	39/35	−192/175	1	−192/245	1/10	−12/35
3	−107/13792	917/3448	−2765/6896	301/431	−6055/3448	1	2765/13792	210/431
4	3/805	−24/575	104/115	−256/115	312/115	−9472/4025	1	24/115

Row i is the formula for y_{n+ki}, k_i = 1/2, 1, 3/2, 2.

Order, Consistency and Zero-Stability

A formula has order p if the coefficients of h^q in its Taylor expansion vanish for q = 0, …, p and the error constant C_{p+1} is non-zero. All seven conditions up to q = 6 hold for every formula and every r, and C_7 is non-zero (Table 3), so the scheme is consistent and of order six. A numerical check agrees: the residual of each formula on $y = e^x$ decreased as $h^{7.0}$ for h = 0.04, 0.02, 0.01 and 0.005, for all three ratios. For zero-stability the block is written $A\, Y_m + B\, Y_{m-1} + C\, Y_{m-2} = 0$ with $Y_m = (y_{n+1/2}, y_{n+1}, y_{n+3/2}, y_{n+2})^T$, and the roots of the first characteristic polynomial π(t, 0) are required to lie in the closed unit disc with those on the circle simple. For r = 1 the polynomial is $\pi(t, 0) = -(8/275\,829\,225)\, t^5\, (t - 1)(8\,561\,213\, t^2 - 3\,181\,816\, t - 549\,397)$, and the roots for all three ratios are in Table 3. In each case t = 1 is simple and the remaining non-zero roots have modulus below one, so the scheme is zero-stable and, being consistent, convergent (Butcher, 2016; Hairer & Wanner, 1996).

Table 3: Order, Error Constants C_7 of the Four Formulae and Non-Zero Roots of π(t, 0) (ρ = 1/2)

r	Order	C_7 (formulae 1–4)	Non-zero roots of π(t, 0)
1	6	-7.53×10^{-4}, 8.67×10^{-4}, 5.76×10^{-4}, -6.83×10^{-4}	1.0000, 0.5000, −0.1283
2	6	-4.02×10^{-4}, 1.83×10^{-3}, 7.85×10^{-1}, -1.28×10^{-3}	1.0000, 0.2075, −0.0005
5/8	6	-3.16×10^{-4}, -1.15×10^{-3}, 9.90×10^{-5}, -4.97×10^{-4}	1.0000, 0.9000, 0.1853

Absolute Stability

The region of absolute stability is the set of H = hλ for which all roots of

$$\pi(t, H) = \det\left[A(H)\, t^2 + B(H)\, t + C\right] = 0 \tag{4}$$

lie in the closed unit disc. For r = 1 it was computed from the spectral radius of the block amplification matrix on a fine grid of the H-plane (Figure 1). The unstable region is bounded and lies in the right half-plane, meeting the real axis on (0, 19.73), except for a thin sliver next to the imaginary axis ($1.2 \lesssim |\mathrm{Im}\, H| \lesssim 2.4$, $\mathrm{Re}\, H \geq -0.04$). The scheme is stable along every ray that makes an angle of at most 88.7° with the negative real axis, tested up to $|H| = 10^4$, and is therefore A(α)-stable with $\alpha \approx 88.7°$.

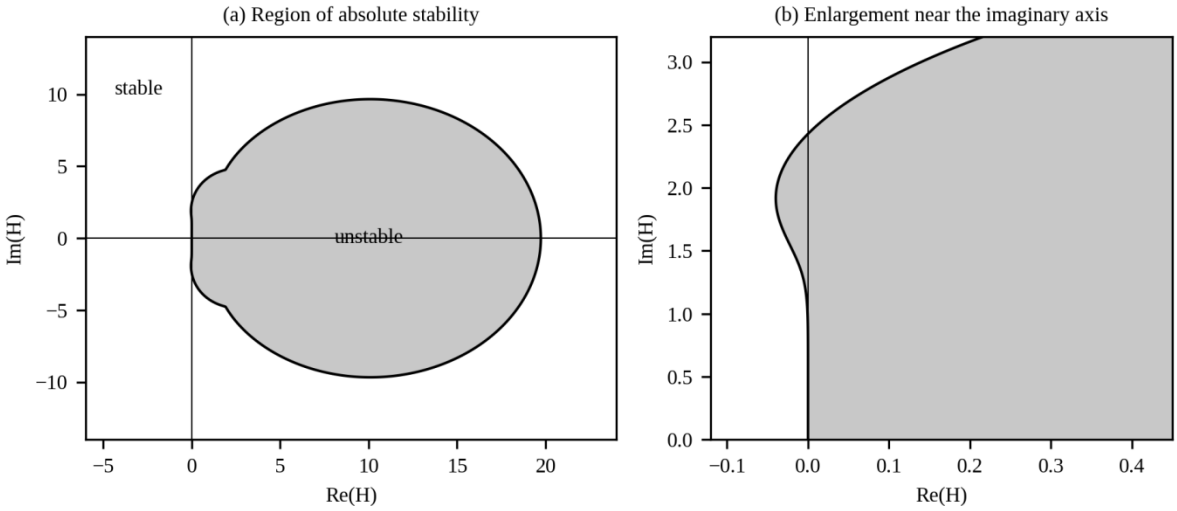


Figure 1: (a) Region of Absolute Stability of HSBBDF for r = 1 and ρ = 1/2; the Shaded Region is Unstable (b) Enlargement Near the Imaginary Axis

For the circuits of Table 1 the eigenvalues lie at 0° (critically and over-damped cases) or 36.9° (under-damped cases) from the negative real axis, well inside the stable wedge. This was checked directly: for every step size used in the runs of Section 5, with both eigenvalues of each circuit, the spectral radius of the r = 1 amplification matrix did not exceed 1 (maximum 0.9992 apart from values equal to 1 within rounding at the smallest steps, where the principal root is close to one). For r = 2 and r = 5/8 the ρ-term refers to the points x_{n+ki-2}, which are not nodes of the preceding block, so a stability region in H alone is not defined without a rule for generating the solution at those points; the stability results above are for the constant-step formula and for zero-stability at all three ratios.

Observed Rate of Convergence on the Circuit Problems

To confirm the order on the actual problems, each circuit was integrated with a constant step (no error control) over a window containing the transient, with $h = 0.2/|\lambda|_{max} \cdot 2^{-m}$, m = 0, …, 3, and the rate was estimated from successive maximum mixed errors. Table 4 shows rates between 5.7 and 6.3 in 17 of the 18 estimates, which agrees with the order established above. The exception is the first rate of the parallel under-damped circuit (4.3), obtained with the coarsest step, where the formula is not yet in its asymptotic range.

Table 4: Observed Rates of Convergence of HSBBDF with a Constant Step

Circuit	Interval	h (largest)	MAXE (largest h)	Observed rates
Series, under-damped	[0, 2]	2.0×10^{-2}	2.9×10^{-6}	5.68, 6.00, 6.04
Series, critically damped	[0, 2]	2.0×10^{-2}	1.2×10^{-5}	6.08, 6.05, 6.03
Series, over-damped	[0, 2]	4.0×10^{-3}	5.4×10^{-7}	6.20, 6.08, 6.04
Parallel, under-damped	[0, 0.2]	2.0×10^{-3}	7.9×10^{-6}	4.28, 6.00, 5.82
Parallel, critically damped	[0, 0.2]	2.0×10^{-3}	1.2×10^{-4}	6.08, 6.05, 6.16
Parallel, over-damped	[0, 0.1]	1.0×10^{-4}	9.9×10^{-7}	6.26, 6.09, 6.05

Variable-Step Implementation

The circuit problems are linear, so each block requires the solution of one linear system of order 4N = 8, and the Newton iteration of the general scheme converges in a single step. The step size is controlled as follows.

Error estimate. For each of the four new points a predictor y^p is obtained by extrapolating the polynomial of degree six through the seven most recent stored nodes, and y^c is the value from the block. The residual of formula i on the exact solution is $C_{7,i}\, h^7\, y^{(7)}$, and the extrapolation error is $P_i\, y^{(7)}$, where P_i is the product of the distances to the seven nodes divided by 7!. Milne's device then gives the local error estimate

$$\varepsilon_i = \frac{C_{7,i}\,h^7}{P_i - C_{7,i}\,h^7}\left(y^c_{n+k_i} - y^p_{n+k_i}\right), \qquad P_i = \frac{1}{7!}\prod_{j=1}^{7}\left(x_{n+k_i} - x_{(j)}\right) \tag{5}$$

where $x_{(j)}$ are the seven nodes. The factor $1/(1 - h\beta_i\lambda)$ that multiplies the local error of an implicit formula is ignored, so the estimate is conservative for stiff problems. A comparison with the true local error, computed from exact back values on several blocks of the parallel under-damped circuit, gave ratios of true error to estimate between 0.25 and 13, so the estimate is of the right size but not exact.

Step control. A block is accepted if $\max_i |\varepsilon_i|/(1 + |y^c_{n+ki}|) \le$ TOL/2, with the maximum taken over the four points and over the components. After an accepted block the step is enlarged by 8/5 (so that r = 5/8 applies to the next block) if the estimate multiplied by $(8/5)^7$ is at most TOL/10, and otherwise it is kept (r = 1). A rejected block is repeated with the step halved; when the failed step equals the preceding accepted step this corresponds to r = 2. In the less common situations in which the ratio between the preceding step and the trial step is not 1, 2 or 5/8 (for example after two successive rejections), the r = 1 formula is used with back values at the new spacing generated by interpolation.

Interpolation. Whenever a value is required at a point that is not a stored node, it is taken from the Lagrange polynomial of degree nine through the ten nearest stored nodes. This is needed for the ρ-term at x_{n+ki-2} when $r \neq 1$, since those points do not coincide with the grid of the preceding block. The cost of this step is not present in the constant-step formula.

Starting values and settings. The back values and function values needed by the first block were taken from the analytical solution, as is usual in this literature, and the initial step was $h_0 = 0.01/|\lambda|_{max}$. All circuits were integrated over

$0 \le t \le 10$, the interval shown in the figures of Zainuddin et al. (2026), with TOL = 10^{-2}, 10^{-4} and 10^{-6}. The accuracy measure is the maximum mixed error over all stored points (two solution points and two off-step points per block) and both components,

$$\mathrm{MAXE} = \max_i \max_w \frac{|y_{w,i} - y_w(t_i)|}{1 + |y_w(t_i)|} \tag{6}$$

the same measure as in the source study. SS is the number of accepted blocks and FS the number of rejected trials. The solver was written in Python 3.12 with NumPy. To obtain a comparison of run times on one machine, the variable-order BDF code of SciPy (Virtanen et al., 2020), called with the exact Jacobian and rtol = atol = TOL, was run as well. It follows the quasi-constant step-size approach used in ode15s (Shampine & Reichelt, 1997) and is used here as a proxy, not as a replacement for ode15s. Run times are the median of 15 repetitions.

RESULTS AND DISCUSSION

Figure 2 shows the HSBBDF solution on the transient window for each circuit at TOL = 10^{-2}; the computed points follow the exact curve closely, including the oscillation of the under-damped cases. The step sizes (Figure 3) are smallest at the start and grow by three to five orders of magnitude as the transient decays, reaching about 1 s at the end of the interval for TOL = 10^{-2}. Rejected trials were rare and appear as short drops in the step-size curves.

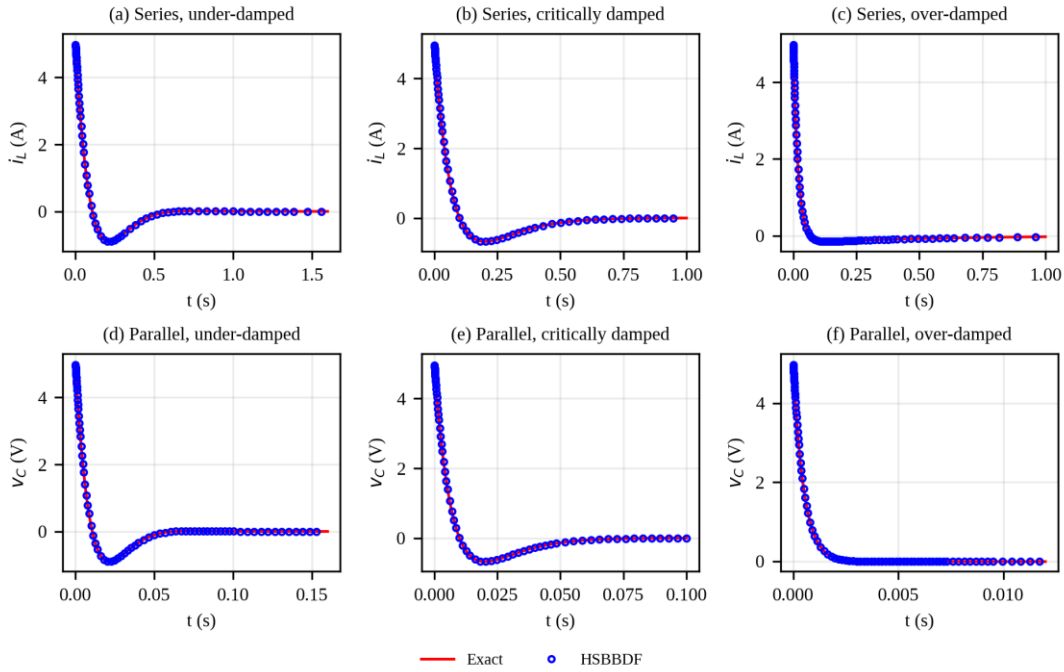


Figure 2: HSBBDF solution (circles) and exact solution (line) for the six circuits at TOL = 10^{-2}; (a)–(c) show the inductor current of the series circuit and (d)–(f) the capacitor voltage of the parallel circuit

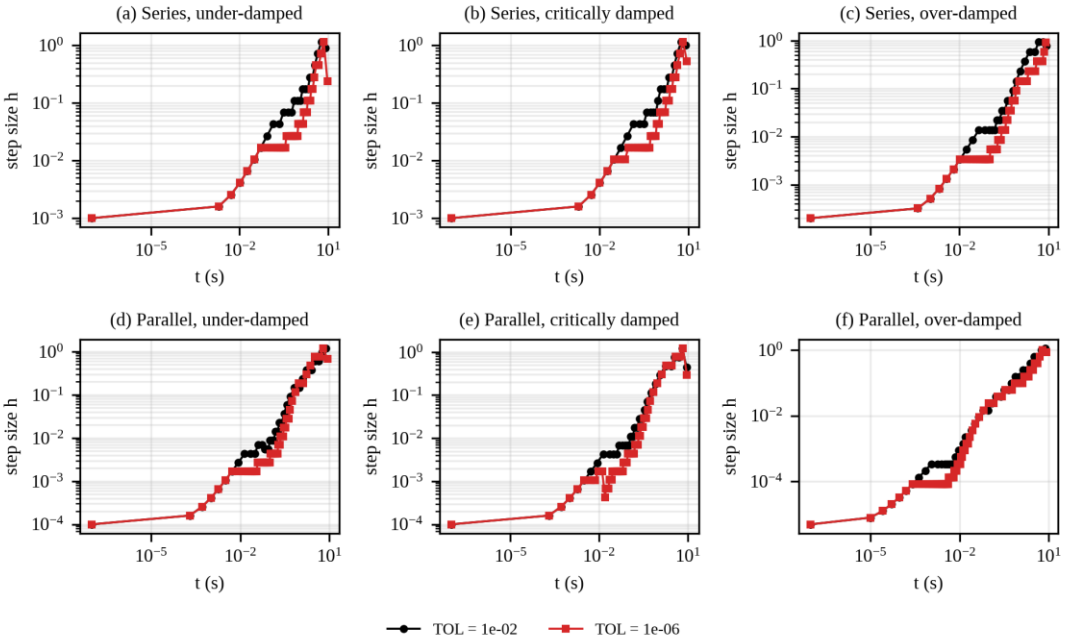


Figure 3: Step size against time at TOL = 10^{-2} and TOL = 10^{-6}

Comparison with VSVO 2DBBDF and ode15s

Tables 5 and 6 give SS, FS and MAXE for HSBBDF and, as reported by Zainuddin et al. (2026), for VSVO 2DBBDF and ode15s. SS counts blocks for HSBBDF and VSVO 2DBBDF (each block yields several solution values) and steps for ode15s.

Table 5: Series RLC Circuits: Successful Steps (SS), Failed Steps (FS) and Maximum Mixed Error (MAXE)

TOL	HSBBDF			VSVO 2DBBDF			ode15s		
	SS	*FS*	*MAXE*	*SS*	*FS*	*MAXE*	*SS*	*FS*	*MAXE*
Series, under-damped									
10^{-2}	25	0	5.58×10^{-4}	61	0	2.12×10^{-3}	46	1	1.73×10^{-2}
10^{-4}	32	0	2.44×10^{-5}	172	2	2.80×10^{-4}	85	1	4.60×10^{-4}
10^{-6}	48	0	7.40×10^{-7}	235	2	5.96×10^{-6}	170	2	2.83×10^{-6}
Series, critically damped									
10^{-2}	25	0	2.52×10^{-4}	67	0	1.25×10^{-3}	48	3	1.25×10^{-2}
10^{-4}	31	0	2.64×10^{-5}	123	0	8.84×10^{-5}	87	1	1.60×10^{-4}
10^{-6}	46	0	1.71×10^{-6}	237	1	1.58×10^{-5}	164	4	2.97×10^{-6}
Series, over-damped									
10^{-2}	29	0	6.02×10^{-4}	72	0	3.08×10^{-3}	54	1	9.37×10^{-3}
10^{-4}	38	0	2.81×10^{-5}	135	0	2.26×10^{-4}	100	0	1.51×10^{-4}
10^{-6}	58	0	3.26×10^{-7}	294	0	1.58×10^{-5}	187	3	2.42×10^{-6}

Table 6: Parallel RLC Circuits: Successful Steps (SS), Failed Steps (FS) and Maximum Mixed Error (MAXE)

TOL	HSBBDF			VSVO 2DBBDF			ode15s		
	SS	*FS*	*MAXE*	*SS*	*FS*	*MAXE*	*SS*	*FS*	*MAXE*
Parallel, under-damped									
10^{-2}	35	1	1.06×10^{-3}	68	0	2.12×10^{-3}	61	2	1.73×10^{-2}
10^{-4}	41	0	1.02×10^{-4}	181	2	2.80×10^{-4}	114	2	4.60×10^{-4}
10^{-6}	59	0	6.02×10^{-6}	244	2	5.96×10^{-6}	193	5	2.83×10^{-6}
Parallel, critically damped									
10^{-2}	34	0	7.67×10^{-4}	74	0	1.25×10^{-3}	56	4	1.25×10^{-2}
10^{-4}	42	0	7.06×10^{-5}	131	0	8.84×10^{-5}	108	1	1.60×10^{-4}
10^{-6}	63	2	1.09×10^{-5}	245	1	1.58×10^{-5}	193	5	2.97×10^{-6}
Parallel, over-damped									
10^{-2}	45	0	2.21×10^{-3}	77	0	4.36×10^{-3}	74	5	9.18×10^{-3}
10^{-4}	54	0	9.01×10^{-5}	137	0	3.33×10^{-4}	126	0	1.60×10^{-4}
10^{-6}	79	0	4.72×10^{-7}	279	0	2.32×10^{-5}	237	3	2.97×10^{-6}

VSVO 2DBBDF and ode15s: values as reported by Zainuddin et al. (2026). HSBBDF: this study.

At TOL = 10^{-2} and 10^{-4}, HSBBDF gave the smallest maximum error in all twelve cases: by factors of 1.6 to 5.1 (10^{-2}) and 1.3 to 11.5 (10^{-4}) relative to VSVO 2DBBDF, and by factors of 4.2 to 50 and 1.8 to 19 relative to ode15s. At TOL = 10^{-6} HSBBDF was more accurate than VSVO 2DBBDF in five of six cases (by up to a factor of 49 for the over-damped circuits) and equal to it within 1% in the sixth (parallel, under-damped: 6.02×10^{-6} against 5.96×10^{-6}). ode15s was more accurate at TOL = 10^{-6} in two cases, the parallel under-damped and parallel critically damped circuits (2.83×10^{-6} against 6.02×10^{-6}, and 2.97×10^{-6} against 1.09×10^{-5}), and HSBBDF was more accurate in the other four, by a factor of up to 7.4. Over all 18 cases HSBBDF was more accurate than VSVO 2DBBDF in 17 and then ode15s in 16.

The number of steps was smaller in all 18 cases: HSBBDF needed 25 to 79 blocks, against 61 to 294 for VSVO 2DBBDF and 46 to 237 steps for ode15s, which is 1.7 to 5.4 and 1.6 to 3.6 times fewer, respectively. Figure 4 shows the accuracy against the number of steps. Interpolating the HSBBDF results in this plane, about 33 to 60 blocks would reach the error that VSVO 2DBBDF attained at TOL = 10^{-6} with 235 to 294 blocks, and about 41 to 69 blocks the error of ode15s with 164 to 237 steps, so the saving is roughly a factor of three to seven; this interpolated figure is an indication only. HSBBDF also had few failed trials: 3 over the 18 runs, against 10 for VSVO 2DBBDF and 43 for ode15s as reported.

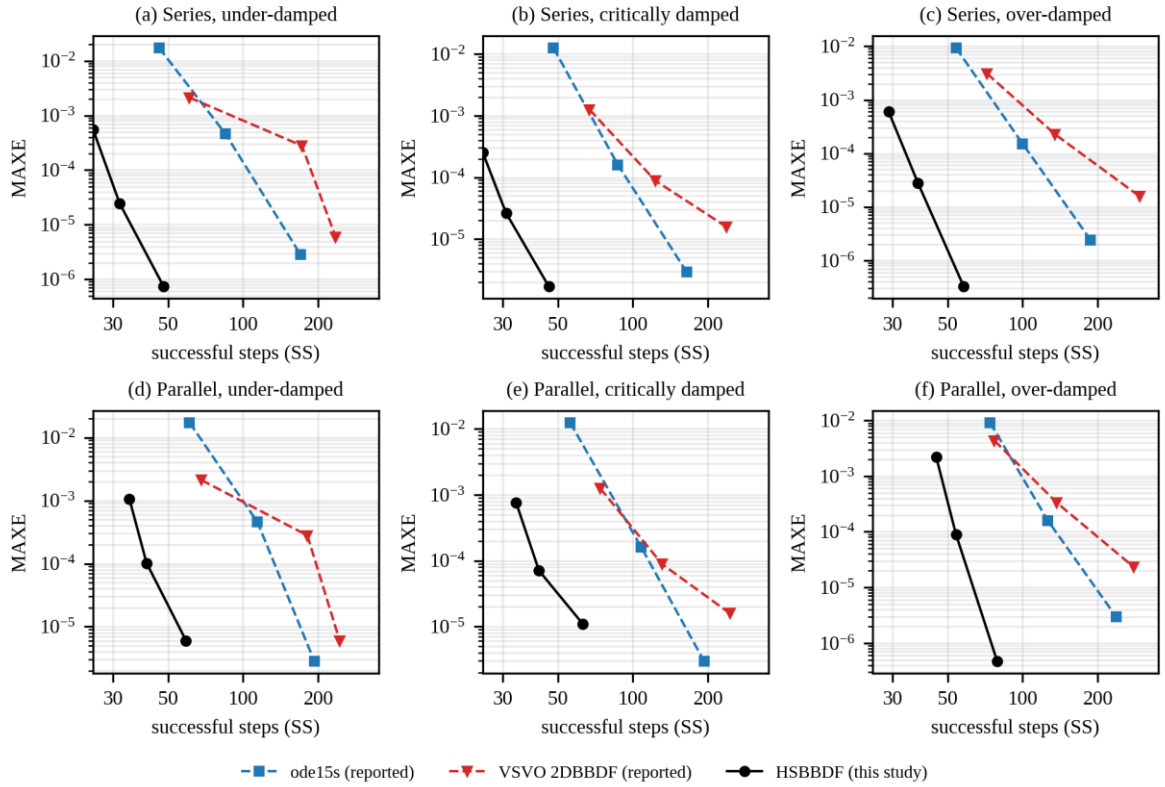


Figure 4: Maximum mixed error against the number of successful steps for the six circuits. VSVO 2DBBDF and ode15s as reported by Zainuddin et al. (2026); the three points of each curve correspond to TOL = 10^{-2}, 10^{-4} and 10^{-6}

The global error is not equal to the tolerance, which applies to the local error estimate. For HSBBDF the ratio MAXE/TOL lies between 0.03 and 10.9 (mean 1.3); for the published results it lies between 0.1 and 23 for VSVO 2DBBDF and between 0.9 and 4.6 for ode15s. Comparisons at the same TOL are therefore not comparisons at the same error, which is why Figure 4 is the more reliable view.

Two cautions apply to the reference values. First, they were not reproduced here, since MATLAB was not used, and some of the published entries coincide across different circuits: the under-damped series and parallel circuits have identical MAXE for both methods at all three tolerances, as do the two critically damped circuits, and ode15s has the same MAXE (1.60×10^{-4} at TOL = 10^{-4}, 2.97×10^{-6} at TOL = 10^{-6}) in three of the six tables. This may be a property of the problems, since the parallel circuits are time-scaled versions of the series ones, but it was not possible to confirm it. Second, run times in the source study were measured on another machine and are not comparable with those measured here, so they are not compared.

Comparison with a Variable-order BDF Code on the Same Platform

Table 7 compares HSBBDF with SciPy's variable-order BDF, both run at the same tolerance on the same machine. HSBBDF used 1.2 to 3.3 times fewer steps (blocks against single steps) and gave a smaller maximum mixed error in all 18 cases, by a factor of 3.5 to 400. Part of this difference reflects the different relation between tolerance and global error of the two codes: the BDF error was 1.4 to 82 times the tolerance. The wall-clock time of HSBBDF was, however, 1.2 to 4.9 times that of SciPy's BDF in every case. Profiling the prototype showed that about two thirds of its run time is spent in the Lagrange routines used for interpolation and for the predictor, so the time reflects the cost of this implementation rather than the number of linear systems solved (one per block). An implementation that keeps divided differences instead of recomputing the interpolation weights would reduce this cost, but this has not been tested.

Table 7: HSBBDF and SciPy BDF Run on the Same Machine: Steps, Maximum Mixed Error and Run Time

Circuit	**TOL**	**HSBBDF**			**BDF (SciPy)**		
		SS	*MAXE*	*Time (ms)*	*Steps*	*MAXE*	*Time (ms)*
Series, under-damped	10^{-2}	25	5.58×10^{-4}	19.2	32	8.00×10^{-2}	4.6
	10^{-4}	32	2.44×10^{-5}	22.8	73	1.91×10^{-3}	9.7
	10^{-6}	48	7.40×10^{-7}	31.1	141	2.83×10^{-5}	18.8
Series, critically damped	10^{-2}	25	2.52×10^{-4}	19.3	37	3.60×10^{-2}	5.3
	10^{-4}	31	2.64×10^{-5}	22.3	81	1.19×10^{-3}	18.3
	10^{-6}	46	1.71×10^{-6}	30.7	152	2.06×10^{-5}	20.8
Series, over-damped	10^{-2}	29	6.02×10^{-4}	23.5	58	1.38×10^{-2}	8.3
	10^{-4}	38	2.81×10^{-5}	29.5	94	2.97×10^{-4}	22.6

Circuit	TOL	HSBBDF			BDF (SciPy)		
		SS	*MAXE*	*Time (ms)*	*Steps*	*MAXE*	*Time (ms)*
	10^{-6}	58	3.26×10^{-7}	37.9	180	1.05×10^{-5}	23.8
Parallel, under-damped	10^{-2}	35	1.06×10^{-3}	28.4	47	7.15×10^{-2}	6.5
	10^{-4}	41	1.02×10^{-4}	30.5	83	4.63×10^{-3}	11.1
	10^{-6}	59	6.02×10^{-6}	39.8	166	6.07×10^{-5}	23.0
Parallel, critically damped	10^{-2}	34	7.67×10^{-4}	25.6	47	3.07×10^{-1}	6.7
	10^{-4}	42	7.06×10^{-5}	30.2	96	3.10×10^{-3}	13.2
	10^{-6}	63	1.09×10^{-5}	46.7	173	8.25×10^{-5}	22.9
Parallel, over-damped	10^{-2}	45	2.21×10^{-3}	35.8	52	1.43×10^{-2}	7.4
	10^{-4}	54	9.01×10^{-5}	37.4	119	3.18×10^{-4}	15.8
	10^{-6}	79	4.72×10^{-7}	51.7	222	1.93×10^{-5}	29.6

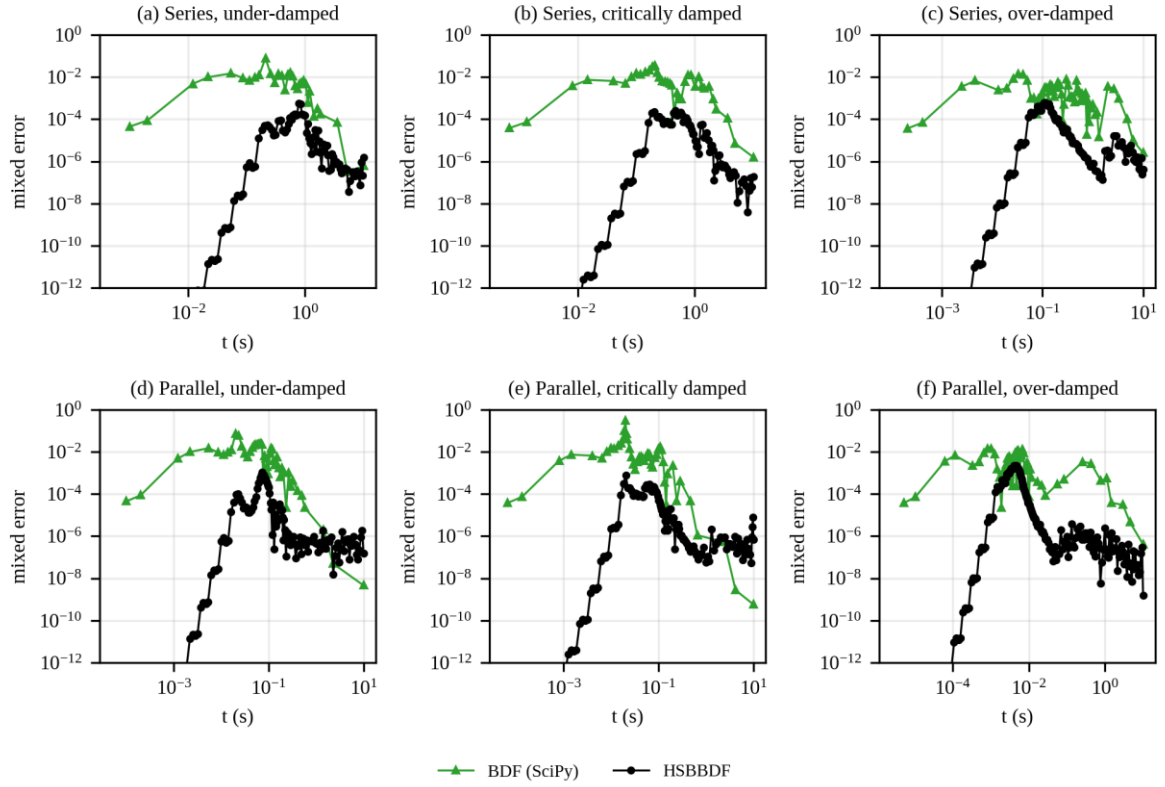


Figure 5: Mixed error at the computed points against time at TOL = 10^{-2} for HSBBDF and SciPy BDF

Figure 5 gives the mixed error along the integration at TOL = 10^{-2}. In every circuit the error of HSBBDF during the transient is smaller than that of the BDF code by one to several orders of magnitude, as the error of an order-six formula would suggest. At late times the BDF error falls below that of HSBBDF in the parallel under-damped and critically damped circuits (to about 10^{-8} against 10^{-6}), where the error of HSBBDF levels off near 10^{-6} to 10^{-5}. The solution has by then decayed to a negligible size and both errors are far below the tolerance. The cause of the levelling was not investigated; it may be related to the fact that HSBBDF is not L-stable.

Conclusion

A higher-order hybrid block BDF with two off-step points was applied to the source-free response of series and parallel RLC circuits. The scheme is of order six, a result supported by constant-step rates of about six on the six circuit problems, zero-stable for the step-size ratios 1, 2 and 5/8, and A(α)-stable with $\alpha \approx 88.7°$ for $r = 1$. With a variable-step implementation it gave a smaller maximum mixed error than the published results of VSVO 2DBBDF in 17 of 18 cases and than ode15s in 16 of 18, using 1.6 to 5.4 times fewer steps, although ode15s was more accurate for two circuits at the tightest tolerance. On the same machine it also needed fewer steps and gave smaller errors than SciPy's variable-order BDF, but its Python prototype was slower in wall-clock time. Hence, the proposed scheme is recommended for solving stiff ODE and simple oscillatory cases.

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